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THE STANDARD (LEVEL) TRIAXIAL ELLIPSOID OF REFERENCE AND THE FORCE OF GRAVITY AT ITS SURFACE

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FOREWARD

The problem concerning the standard triaxial ellipsoid, which is closely connected with the theory of the figures of equilibrium, is the problem to be analyzed here, since the triaxility of our planet Earth has been definitely proved in extremely diverse ways. (By triaxiality we imply here the ellipsoid that best approximates the actual standard, or level, surface of the Earth, that is, the so-called geoid.)

By virtue of this it is expedient to introduce the solution of Stokes' problem not for an ellipsoid of revolution, for which this problem is fully solved, but for a triaxial ellipsoid. The fact is, for a standard triaxial ellipsoid the solution of Stokes's problem, strangely enough, is still not completed by means of the mathematical device most essential in the given problem, namely the device of Lame's functions. Therefore, the solution which we shall give here is carried out precisely with the aid of these functions. The somewhat minor popularity of these functions in problems of theoretical gravimetry causes the author to linger long enough to give sufficient details regarding these functions. The rigorous solution obtained for Stokes' problem enables one to give for the triaxial Earth a generalized Clairaut formula, which finds application in the determination of compression in an ellipsoid. The various data on the triaxility of the Earth is compiled in the annex.

Leningrad, 1946
D. Zagrebin

INTRODUCTION

With the discovery of Newton's law there immediately appeared two basic problems of celestial mechanics: the problem of determining the motion of two or more material points whose interaction is controlled by the forces of Newtonian attraction and the problem of determining the form of a continuous rotating liquid mass, homogeneous or not, whose particles are acted upon by the same Newtonian attractions.

By their works on these two problems, prominent mathematicians and physicists--Clairaut, Maclaurin, Jacobi, Poinçare, Lyapunov, and others--developed the theory of the figures of equilibrium.

This unified theory can be analyzed into two sharply divergent classes. (1) the theory of figures of equilibrium for a rotating homogeneous fluid; and (2) the theory of figures of equilibrium of a rotating nonhomogeneous fluid mass (Clairaut's problem); to this same class is related the theory of ellipsoidal figures of equilibrium of a rotating fluid without any assumptions concerning the internal distribution of density in the mass (Stokes' problem).

Relative to the first class of this theory it is necessary to state that although the theory of figures of equilibrium of a rotating homogeneous fluid does not have immediate application in the problem concerning the figures of planets, which are nonhomogeneous, all of this theory was thoroughly studied by some of the greatest mathematicians.

Naturally, it is to be expected that the figure of equilibrium of a rotating homogeneous fluid mass has as its axis of symmetry the axis of rotation; the first result of this theory was the so-called ellipsoid of Maclaurin (diaxial).

However, in 1834, Jacobi showed for the first time that even a triaxial ellipsoid could also be a figure of equilibrium. This figure of equilibrium possesses three planes, or levels, of symmetry, but for this case the axis of rotation is no longer the axis of symmetry (cf Jacobi C. G. J. Ueber die Figur des Gleichgewichts. Ann. Phys. und Chem. Herausgegeben zu Berlin von J. C. Poggendorff. XXXIII, 1834).

Later, in 1885, Poincare showed that for small variations in a certain parameter the configuration of equilibrium gives way to another adjacent configuration of equilibrium. For further variations of this parameter there is obtained a series of figures of equilibrium which Poincare called "linear series." Thus, we have a series of Maclaurin ellipsoids and Jacobi ellipsoids; it is interesting to note here that for a transition from one of these series to another, at a certain critical point corresponding to a so-called bifigurational ellipsoid, there occurs a shift in the stability of these figures. This ellipsoid of bifiguration is thus simultaneously both a Maclaurin ellipsoid and a Jacobi ellipsoid (cf Poincare, H. Sur l'equilibre d'une masse fluide animee d'un mouvement de rotation. Acta Math. 7; 1885).

Equilibrium in Maclaurin ellipsoids is expressed in Lamé's functions thus:

$$\frac{\omega^2}{T} = \frac{R_2 S_2 - R_1 S_1}{R_2^2}$$

where omega is the angular velocity of rotation, T is the volume of the ellipsoid, and R, S are Lame functions of the first and second kind in Poincare's notation as used by him in his book, Figures d'equilibre d'une massé fluide.

For Jacobi ellipsoids it is necessary that one more condition be satisfied:

$$\frac{R_3 S_3 - R_1 S_1}{R_3^2} = \frac{R_2 S_2 - R_1 S_1}{R_2^2}$$

Lyapunov showed that if for certain conditions, besides ellipsoids of revolution, a triaxial ellipsoid can also have a figure of equilibrium, then the latter ellipsoid is stable (cf Lyapunov A. Ob Ustoychivosti Ellipsoidal'nykh Form Ravnovesiya Vrashchayushcheysya Zhidkosti [Stability of Ellipsoidal Figures of Equilibrium of Rotating Fluids] SPb. 1834).

The second class of this theory plays a large role in the theory of the figure of the Earth; that is why this theory is now regarded as a part of physical geodesy.

Up to the middle of the past century, more precisely up to the works of Stokes (1849), the theory of the figure of the Earth, initiated by Clairaut, was developed on the basis of certain assumptions concerning the internal structure of the Earth (cf Stokes G. On Attractions and Clairaut's Theorem. Cambridge and Dublin, Math. J. 4, 1849; Clairaut A. Theorie de la figure de la terre, tiree de principes de l'hypostatique. 1743).

In his class work of 1743, Clairaut sought, as the figure of equilibrium of a nonhomogeneous slowly rotating fluid, figures closely approximating a sphere. As a result of this work, there appeared the so-called spheroid of Clairaut, that is, the standard surface or level of reference, which is determined by the well-known equation $r = a(1-\alpha \sin^2\phi)$ where r is the radius of the vector of this surface, a is the radius of the equator, alpha is the compression parameter, and phi is the geographic latitude. (Strictly speaking, it is necessary here to extend the geocentric latitude, but in the first approximation, just as in Clairaut's investigation, the geocentric latitude can be taken for the geographic latitude, with an accuracy up to quantities of the order of compression.)

The gravitational force on Clairaut's sphere is determined by the following formula: $\gamma = \gamma_e\,(1+\beta \sin^2\phi)$ where γ_e is the gravitational force at the equator and the coefficient beta is the relative surplus of gravity at the poles in comparison with gravity at the equator.

Actually, for ϕ = 90 degrees we obtain from this equation the gravitational force at the poles: $\gamma_p = \gamma_e(1+\beta)$ where $\beta = \frac{\gamma_p - \gamma_e}{\gamma_e}$.

Clairaut's theory established the following relation between this quantity beta and the compression alpha: $\alpha+\beta = \frac{5}{2}m$ where m is the ratio of the centrifugal force to the gravitational force at the equator; that is: $m = \frac{\omega^2 a}{\gamma_e}$. The relation written is accurate up to a quantity of the order of compression (alpha).

Introducing into the expansion for the potential the spherical functions of the second and fourth order, Helmert determined his spheroid (cf Helmert F. Die matematischen und physicialischen Theorien der Höhreren Geodaesie. 1884).

The expression for the gravitational force on this spheroid can be stated in the form:

$$\gamma = \gamma_e\left[1+\beta \sin^2\phi - \beta_1 \sin^2 2\phi + r \cos^2\phi \cdot \cos 2(\lambda - \lambda_a)\right]$$

plus terms of higher order where λ is the geographic longitude and λ_a is the longitude of the great pole, β_1 and r are certain coefficients.

If it is assumed that the equatorial moments of inertia are equal, then the formula takes the form:

$$\gamma = \gamma_e(1+\beta \sin^2\phi - \beta_1 \sin^2 2\phi).$$

Clairaut's theorem for this case is given by the formula which is accurate up to quantities of the second order in alpha inclusively:

$$\alpha+\beta = \frac{5}{2}m - \alpha\left(\alpha+\frac{m}{2}\right) + \frac{2}{7}\delta$$ plus

higher order terms where delta is a certain quantity which Helmert assumed to be determined from observations. However, as noted by Idel'son, these attempts were not successful (Idel'son N. I. Teoriya potentsiala. ONTI, 1936).

As for the determination of this quantity from the assumption that the surface of a spheroid with the assumed accuracy coincides with the surface of an ellipsoid of revolution, this problem is essentially concerned with the transition to the theory developed by Somigliana, which will be discussed later.

The classic work of Clairaut was finished by Roche, Tisserand, Radand, Poincare, and Callandreau. The object of their works was to find agreement between theory and observations; the theory in this case was related in one or another way with the law of densities.

On the base of their investigations, they successfully obtained such a law of densities which gave the necessary agreement between theory and observations.

After the work of Stokes cited above, there appeared the possibility of rejecting all assumptions, whatever they might be, regarding the internal structure of the Earth. In this work Stoke said:

"If, however, we simply accept, as the result of observation, the fact that the terrestrial surface is a surface of equilibrium (we disregard the insignificant inaccuracies of the surface), that is, that it is perpendicular to the direction of gravity, then, independently of any special hypothesis concerning the internal state or any theory except the theory of universal attraction, there exists a necessary connection between the superficial form and the variations in gravity along the surface, the connection being such that if the form is given then the gravitational variation follows."

Here Stokes, in connection with another proof of Clairaut's theorem, stated for the first time the theorem subsequently called by his name.

This theorem Pizzetti has formulated as follows:

"If there is given an external closed level surface, and the planet's mass and angular velocity of rotation are known, then the magnitude and direction of gravity are determined uniquely for each point of the given surface and in all external space" (cf Pizzetti, Paolo. Principii della teoria meccanica della figura dei pianeti. 1913).

This theorem demonstrates the uniqueness, or single-valueness, of the determination (for the given conditions) of the potential function in the external region, and at the same time the uniqueness of the magnitude and direction of gravity. Therefore, in order to solve the problem concerning the gravitational force on the level, or standard, surface, it is necessary first of all to find the external potential function. The search for this function in the case of any planet bounded by a certain level, or standard, surface, with a given mass and angular velocity of rotation, Pizzetti called Stokes' Problem. (N. I. Idel'son includes under Stokes's problem another problem, namely, the determination of the surface of a geoid from observations of the geoid's gravity.)

Thus, in order to solve the problem concerning the gravity on a standard, or level, triaxial ellipsoid we must solve Stokes's problem for this partial case of a level surface. If now we note that the terrestrial surface can be taken as surface can be taken as surface of equilibrium, since as was shown by Stokes, the elevation of the land over the sea is insignificant in comparison with the wide continents, and, in addition, this surface, as we saw earlier, closely approximates a triaxial ellipsoid, then the standard triaxial ellipsoid being discussed by us can be taken as a first approximation to the external standared surface of the Earth, that is, and surface of a geoid.

Somigliana calls the standard ellipsoid thus introduced the ellipsoidal geoid (cf Somigliana C. Sul campo gravitazionale esterno del geoide ellissoidico. Atti della Reale Acc. Nat. dei Lincei (Rendiconti) Ser. Sesta, 11, fasc. 3).

The first one to solve the problem in this way was Hamy in 1890. It is interesting to note that at one stroke he solved the problem (Stokes') with the aid of Lame's functions for the case of a triaxial ellipsoid (cf Hamy. Remarques sur la theorie generale de la figure des planetes. J. de Math. (Liouville) ser. 4,6, 1890).

However, he did not carry out his calculations all the way to the end. After expanding the potential function for the triaxial case he passed over to the case of an ellipsoid of revolution, and he did not give the gravity formula.

Pizzetti solved Stokes' problem for the case of the triaxial ellipsoid with the aid of functions introduced by Morera. As a result he obtained an expression for the components of gravitational force, γ_x, γ_y, γ_z with respect to the coordinate axes.

However, he did not give the formulas for gravity on a triaxial standard, or level, ellipsoid. He merely limited himself to a discussion of the case where this ellipsoid differs slightly from a sphere. Making use of Pizzetti's solution, in 1928 Mineo for the first time gave a closed formula for the gravity on a triaxial level ellipsoid. This formula, which we also shall obtain but only in our own way, has the following form:

$$\gamma = \frac{a\gamma_a \cos^2\phi \cos^2\lambda + b\gamma_b \cos^2\phi \sin^2\lambda + c\gamma_c \sin^2\phi}{\sqrt{(a^2\cos^2\lambda + b^2\sin^2\lambda)\cdot\cos^2\phi + c^2\sin^2\phi}}$$

where a, b, c are the semiaxes of the ellipsoid; γ_a, γ_b, γ_c are the values for gravity at the ends of the semiaxes; and ϕ and λ are the geographical latitude and longitude (cf Mineo C. Sulla gravita superficiale d'un pianeta supposto elissoidico a tre assi. Boll. dell' Unione Mat. ital., Anno VII, No 2, April, 1928).

This formula was obtained also by N. K. Migal' in 1937 (cf Migal' N. Conclusions of the Exact Formula for the Acceleration of Gravity on a Level Surface Having the Figure of a Triaxial Ellipsoid (in Russian). Astro. J. 14, No 5/6, 1937).

By a somewhat different means from ours, Somigliana obtained this same formula (cf Somigliana C. Les expressions finies de la pesanteur normale. Bull. geodes., No 38, 1933). He gave it in a more general form, namely:

$$\gamma = \frac{[(a^2+u)^{\frac{1}{2}}\gamma_a(u)\cos^2\lambda + (b^2+u)^{\frac{1}{2}}\gamma_b(u)\sin^2\lambda]\cos^2\phi + (c^2+u)^{\frac{1}{2}}\gamma_c(u)\sin^2\phi}{\sqrt{[(a^2+u)\cos^2\lambda + (b^2+u)\sin^2\lambda]\cos^2\phi + (c^2+u)\sin^2\phi}}$$

where $\gamma_a(u)$, $\gamma_b(u)$ and $\gamma_c(u)$ are values of the graviation force for the ends of the semiaxes of an ellipsoid, confocal with an ellipsoidal geoid; ϕ and γ are geographical latitude and longitude on an ellipsoid whose equation is

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$$

In this equation u represents a parameter which varies from 0 to infinity.

For the special case u = 0 we obtain Mineo's formula. From this, by the way, Somigliana gives the following general conclusion: "The law governing the distribution of the normal component of the gravitation force is identical for all ellipsoids confocal with an ellipsoidal geoid.

For the case of an ellipsoid of revolution (a = b) this formula thus takes the following form:

$$\gamma = \frac{(a^2+u)^{\frac{1}{2}}\gamma_e(u)\cos^2\phi + (c^2+u)^{\frac{1}{2}}\gamma_p(u)\sin^2\phi}{\sqrt{(a^2+u)\cos^2\phi + (c^2+u)\sin^2\phi}} = \frac{a\gamma_e\cos^2\phi + c\gamma_p\sin^2\phi}{\sqrt{a^2\cos^2\phi + c^2\sin^2\phi}}$$

Somigliana was also first to give these closed expressions for the force of gravity on a level ellipsoid of revolution. According to the solution of the Section on Geodesy of the International Geophysical and Geodesic Union, held in Stockholm in 1930, the normal force of gravity is expressed by the preceding closed formula introduced by us, namely Somigliana's formula, which gives the exact force of gravity on the international ellipsoid. For practical convenience in using this finite formula, Cassinis effected its series expansion, and also gave tables of the normal value of the force of gravity on the international ellipsoid (cf Cassinis G. Sur L'adoption d'une formule internationale pour la pesanteur normale. Bull. geodes., No 26, 1930).

The series-expansion of Somigliana's formula is:

$$\gamma = \gamma_e(1+\beta\sin^2\phi - \beta_1\sin^2 2\phi - \beta_2\sin^2\phi\sin^2 2\phi - \beta_3\sin^4\phi\sin^2 2\phi - \ldots)$$

where β is the relative excess of the force of gravity on the pole in comparison with the equator; γ_e is the force of gravity at the equator. The coefficients β_1, β_2 are expressed in terms of the compression parameter α and quantity β in the following manner:

$$\beta_1 = \frac{1}{2}\alpha(\alpha+2\beta)$$

$$\beta_2 = \frac{1}{8}\alpha^2(2\alpha+3\beta) - \frac{1}{32}\alpha^3(3\alpha+4\beta)$$

$$\ldots\ldots$$

A strictly exact expression of Clairaut's theorem for an ellipsoidal geoid of revolution has the form:

$$\alpha+\beta = \frac{5}{2}m\cdot\chi(\alpha)$$

where $m = \frac{\omega^2 a}{\gamma_e}$ and the chi function $\chi(\alpha)$, which Somigliana introduced, is expressed by the following series:

$$\chi(\alpha) = 1 - \frac{17}{35}\alpha - \frac{1}{245}\alpha^2 - \frac{13}{18865}\alpha^3 - \ldots$$

Elsewhere, instead of this series, they give a series dependent upon the second eccentricity:

$$i^2 = \frac{a^2-c^2}{c^2} = \alpha\cdot\frac{2-\alpha}{(1-\alpha)^2}$$

This series, which is generally designed by C (i), is connected with the preceding chi function $\chi(\alpha)$ by the relation:

$$\chi(\alpha) = \frac{4}{5}(1-\alpha)\cdot C(i^2)$$

It is necessary to say that, in order to solve Stokes' problem for the case of ellipsoidal level surfaces, Lame's functions are most essential. It was also Pizzetti who indicated the possibility of solving this problem by means of these Lame functions (Pizzetti's "Principii della teoria meccanica della figura dei pianeti" (1913) was translated into Russian by A. A. Mikhaylova in 1933.) However, Pizzetti considers this method, first used by Hamy, to be rather tiresome.

The solution of Stokes' problem with Lame's functions for the case of a level ellipsoid of revolution was given by N. I. Idel'son. Rudzki's work was devoted to this problem (cf Rudzki M. Physik der Erde (Leipzig) 1911).

In conclusion, let us remark that, from the general hydrodynamic point of view, all the above-listed problems are limited to a discussion of the simplest case, namely the case of a solid of revolution. In this case, the potential distribution of accelerations in a fluid mass gives equilibrial rotation around an immobile axis (Poincare - Appel theorem); and when the potential of mass forces exists, the hydrodynamic equations for such conditions assume the form:

$$V + \Omega = \text{constant}$$

where V is the potential of mass forces; and Ω is the potential of accelera-tion or, as it is otherwise called, the potential of rotation.

We have touched upon all these problems with the aim of determining the place which our theory assumes among the above-mentioned theories.

- E N D -