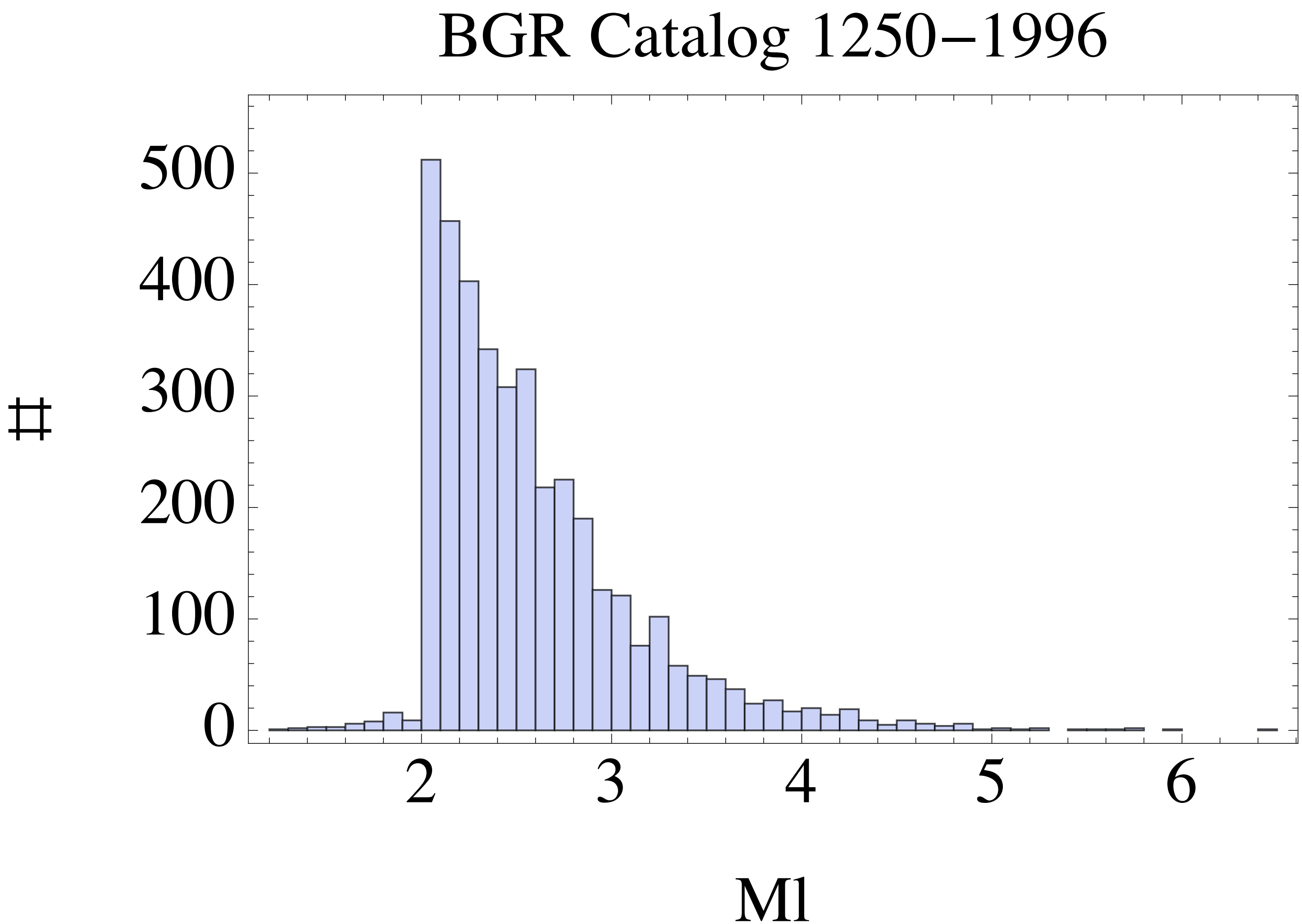
# Crash Course in Probabilistic Seismic Hazard Analysis (PSHA)

**Part 2**

Frank Scherbaum

# From two to many earthquake sources at fixed distance

# Some earthquake statistics



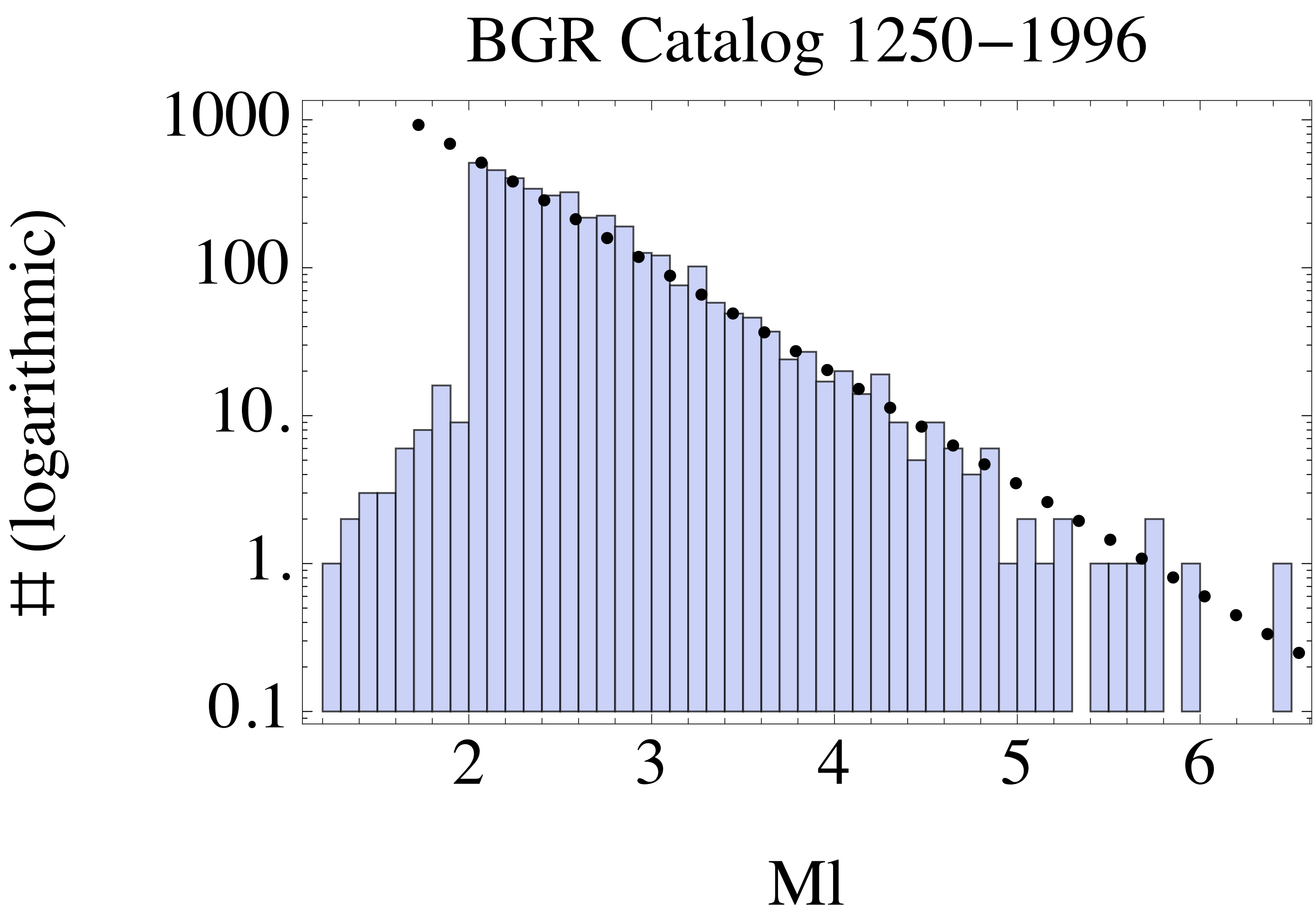
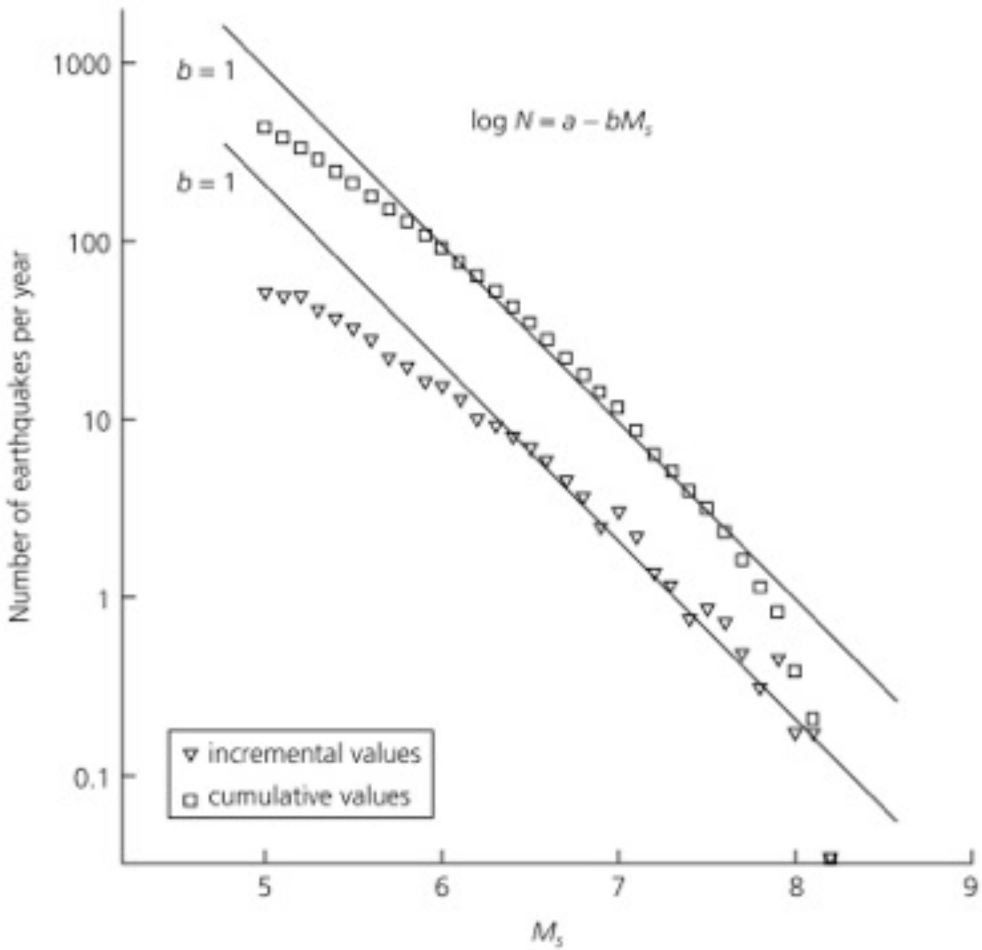
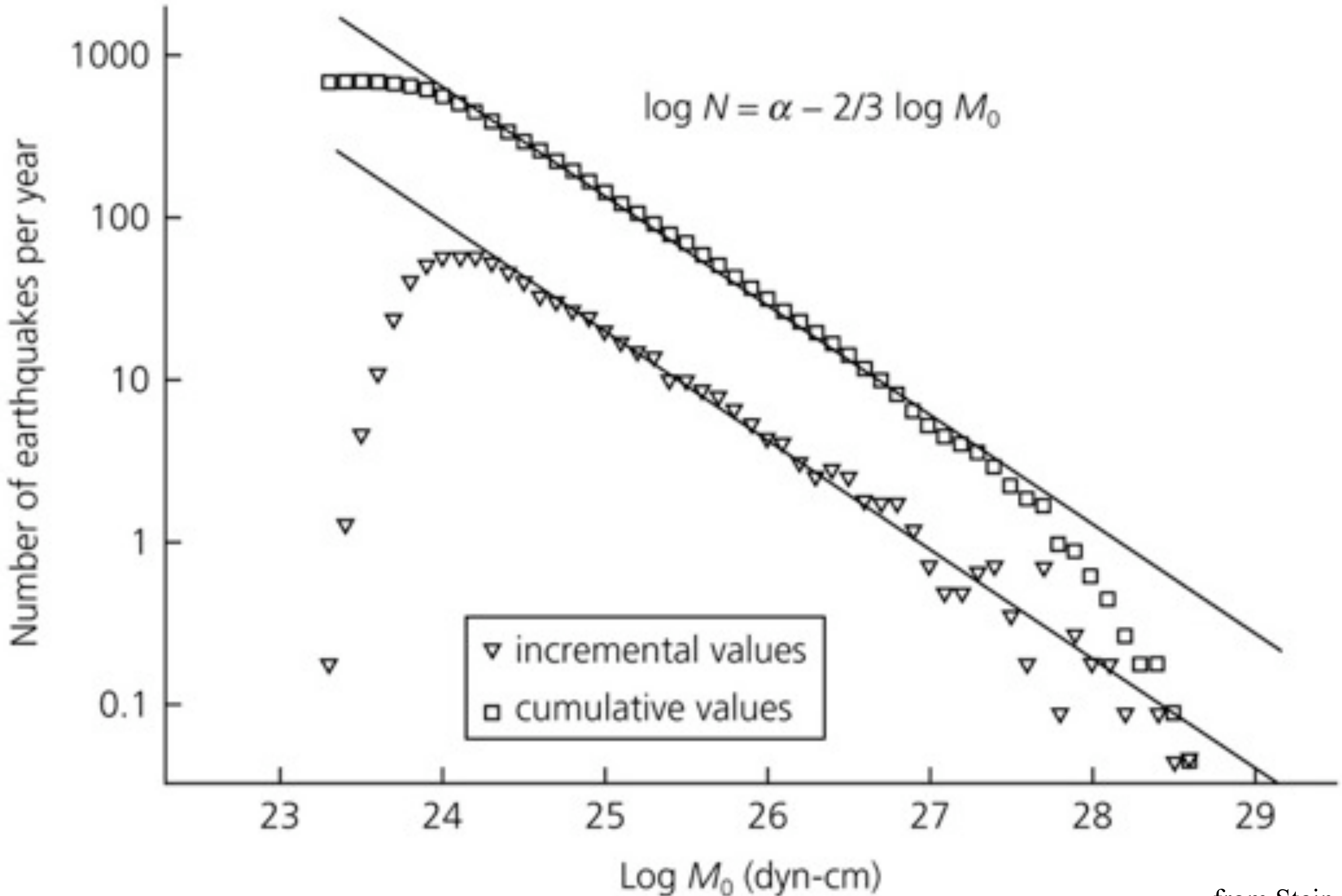

BGR Catalog 1250–1996
1000
100
10.
1.
0.1
# (logarithmic)
2
3
4
5
6
Ml

**Figure 4.7-1: Frequency-magnitude plot for earthquakes during 1968-1997.**




from Stein & Wysession (2003)

**Figure 4.7-2: Frequency-moment plot for earthquakes during 1976-1998.**




from Stein & Wysession (2003)

# Gutenberg-Richter magnitude frequency relation: incremental version

Gutenberg and Richter (1954): $\mathrm{Log}_{10}(N_{M\pm\Delta M/2}) = a - b \cdot M$

$$N_{M\pm\Delta M/2} = 10^{a-b\cdot M} = 10^{a}10^{-b\cdot M} = \alpha \cdot e^{-\beta\cdot M} \quad \text{with} \quad \begin{cases} \alpha = 10^{a} \\ \beta = \ln(10)\cdot b \end{cases}$$

The number of earthquakes as a function of magnitude follows an exponential distribution

a = measure of activity level

b = „b-value“, ratio of large to small earthquakes, measure of degree of fracturing, (stress?), $0.4 < b < 1.8$, global average roughly equal to 1

# Gutenberg-Richter magnitude frequency relation: cumulative version

$$N_{M \pm \Delta M/2} = \alpha \cdot e^{-\beta \cdot M} \quad \Rightarrow \quad N_{\geq M} = ?$$

incremental          cumulative

$$N_{\geq M} = \alpha \int_M^{\infty} e^{-\beta \cdot m} dm = \frac{\alpha}{-\beta} \cdot e^{-\beta \cdot m} \Big|_M^{\infty} = \frac{\alpha}{\beta} \cdot e^{-\beta \cdot M}$$

$$= \alpha_{cum} \cdot e^{-\beta \cdot M} = 10^{a_{cum} - b \cdot M} \quad \text{with} \quad \begin{cases} \alpha_{cum} = \alpha / \beta \\ a_{cum} = a + Log_{10}(e) \ln(\beta) \\ b = \beta / \ln(10) \end{cases}$$

⇒ The cumulative number of earthquakes as a function of magnitude follows an exponential distribution as well

$$Log_{10}(N_{\geq M}) = a_{cum} - b \cdot M$$

# On global scale approximately

| Earthquake number | Magnitude |
|---|---|
| **1** | > 8 |
| **10** | 7 – 7.9 |
| **100** | 6 – 6.9 |
| **1000** | 5 – 5.9 |
| **10000** | 4 – 4.9 |
| **100000** | 3 – 3.9 |

# Truncation

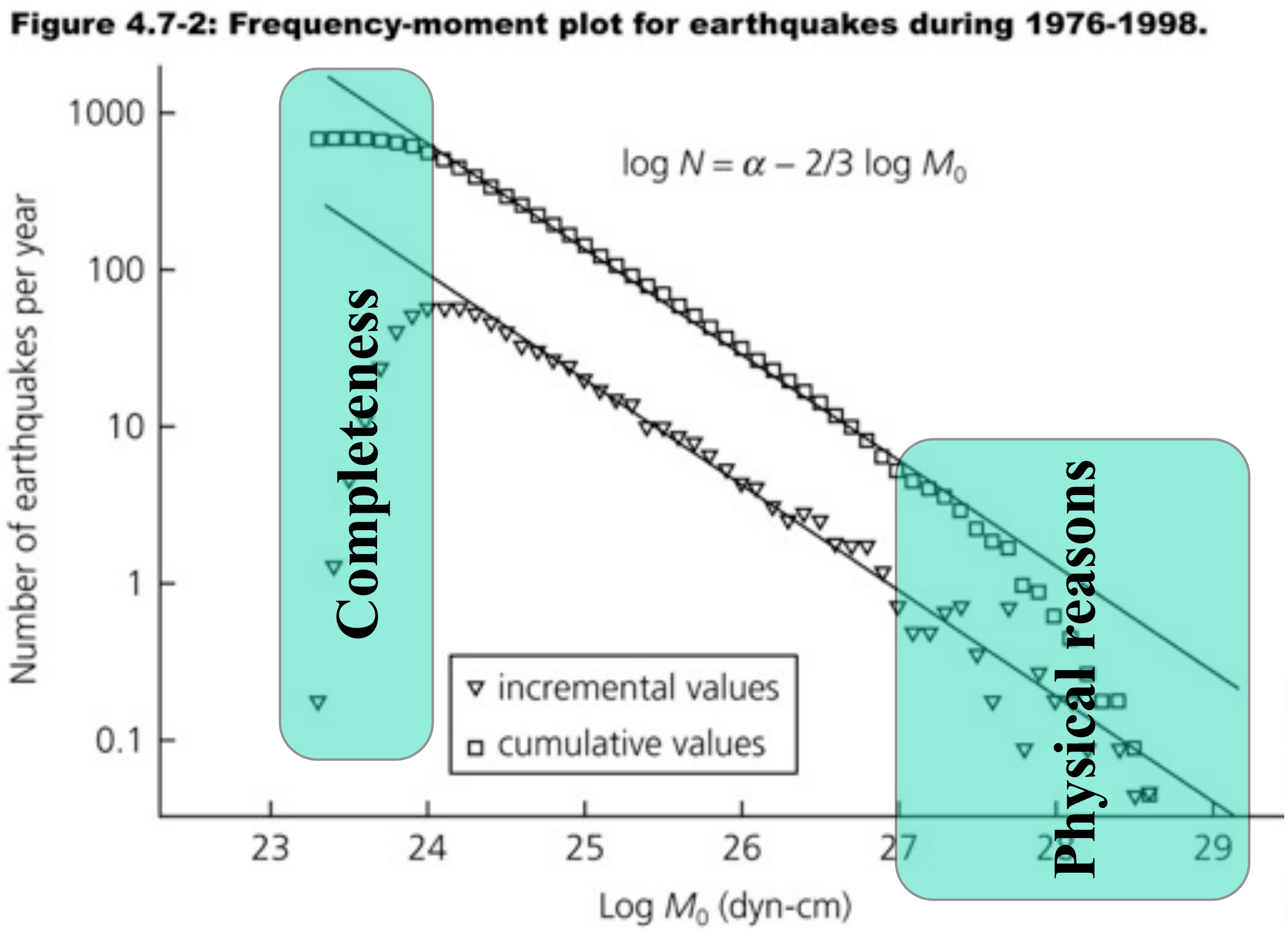
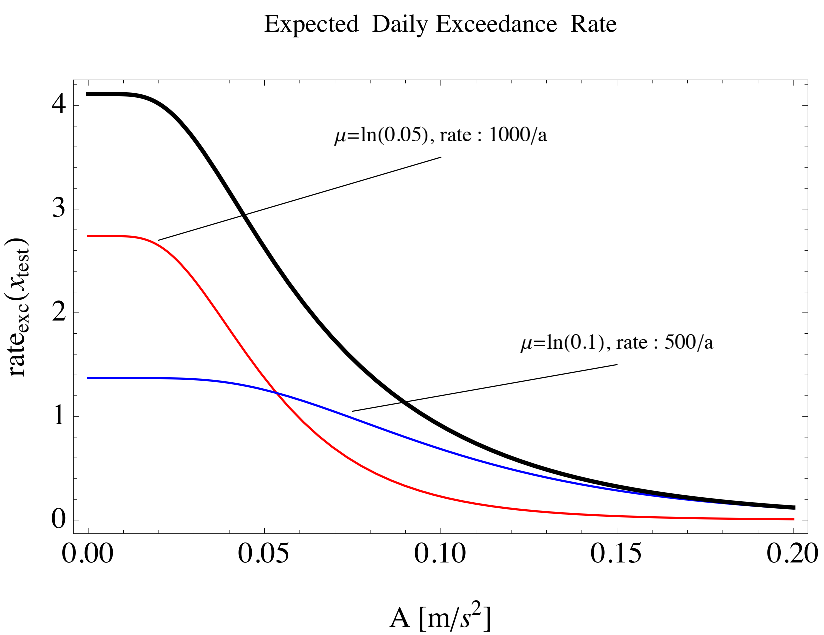


Figure 4.7-2: Frequency-moment plot for earthquakes during 1976-1998.
log N = α – 2/3 log M0
Number of earthquakes per year
1000
100
10
1
0.1
23
24
25
26
27
29
Log M0 (dyn-cm)
incremental values
cumulative values
Completeness
Physical reasons


END
DETOUR

# Goal today

Two sources at fixed distance

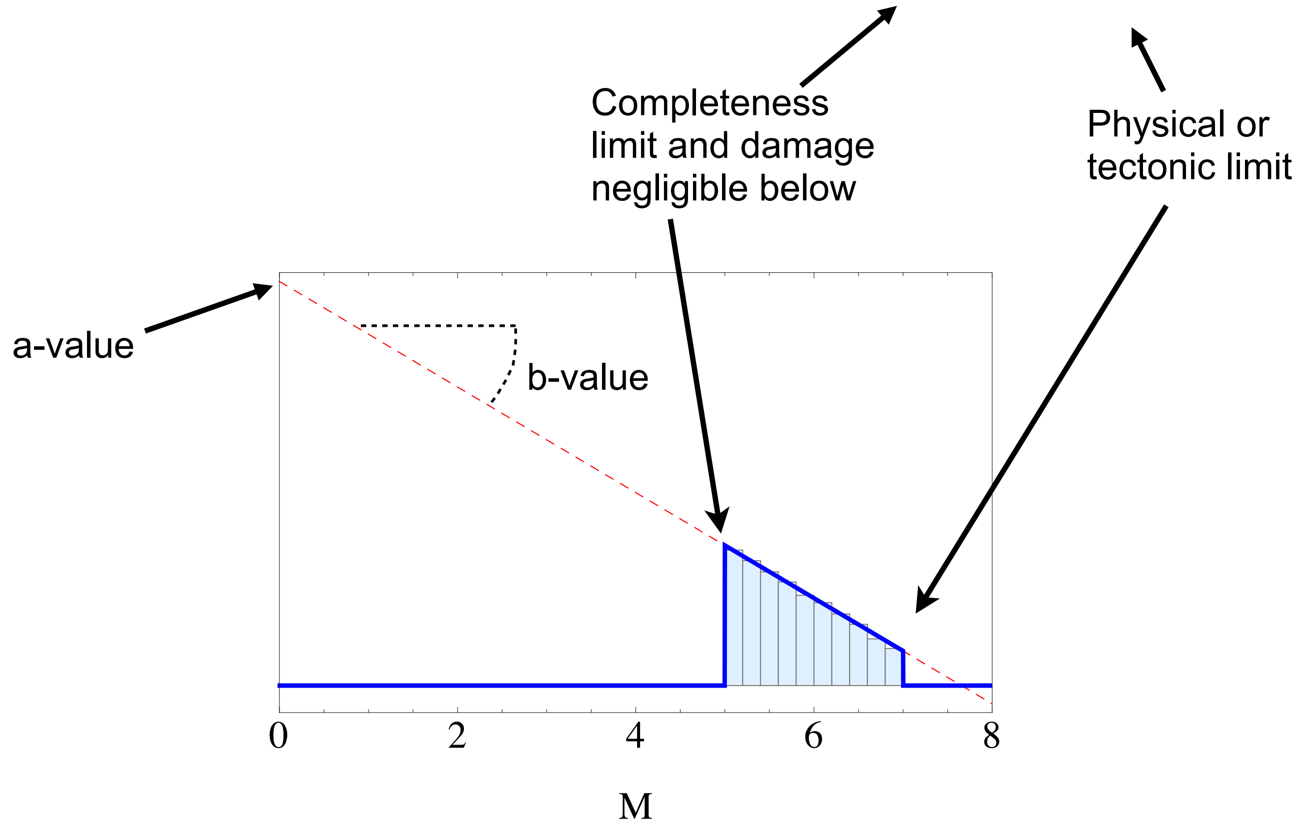



⇒ Distribution of sources at a distribution of distances

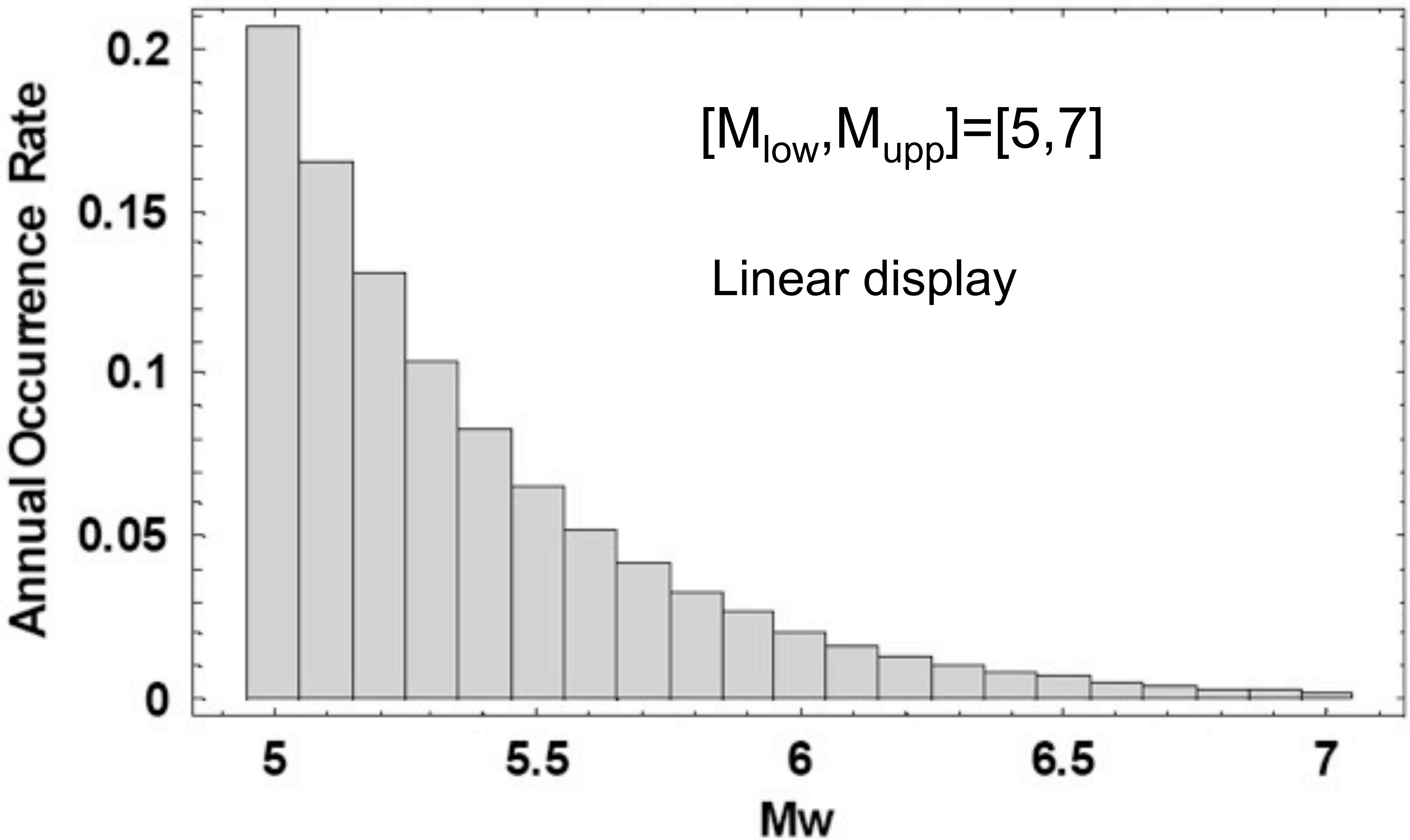
# Earthquake occurrence model

**In seismic hazard analysis:**

Most popular: **doubly truncated exponential** with $[M_{low}, M_{upp}]$



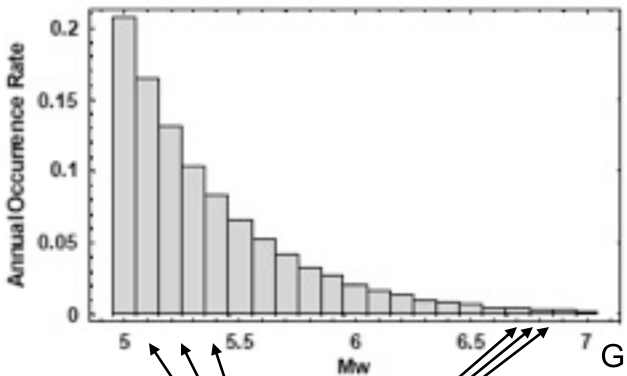
# Doubly truncated magnitude-frequency distribution




$[M_{low},M_{upp}]=[5,7]$

Linear display

# Many source at fixed distance




..............

Ground motion at the site of interest calulated using the model of Berge-Thierry et al. (2003) with 15 km distance

Each bin treated as individual source at the same distance with different occurrence rates

## Principle:

1) Calculate conditional exceedance probability for ground motion level of interest for each magnitude bin:
   $P(x>x_{test}$ | seismic event in magnitude bin)

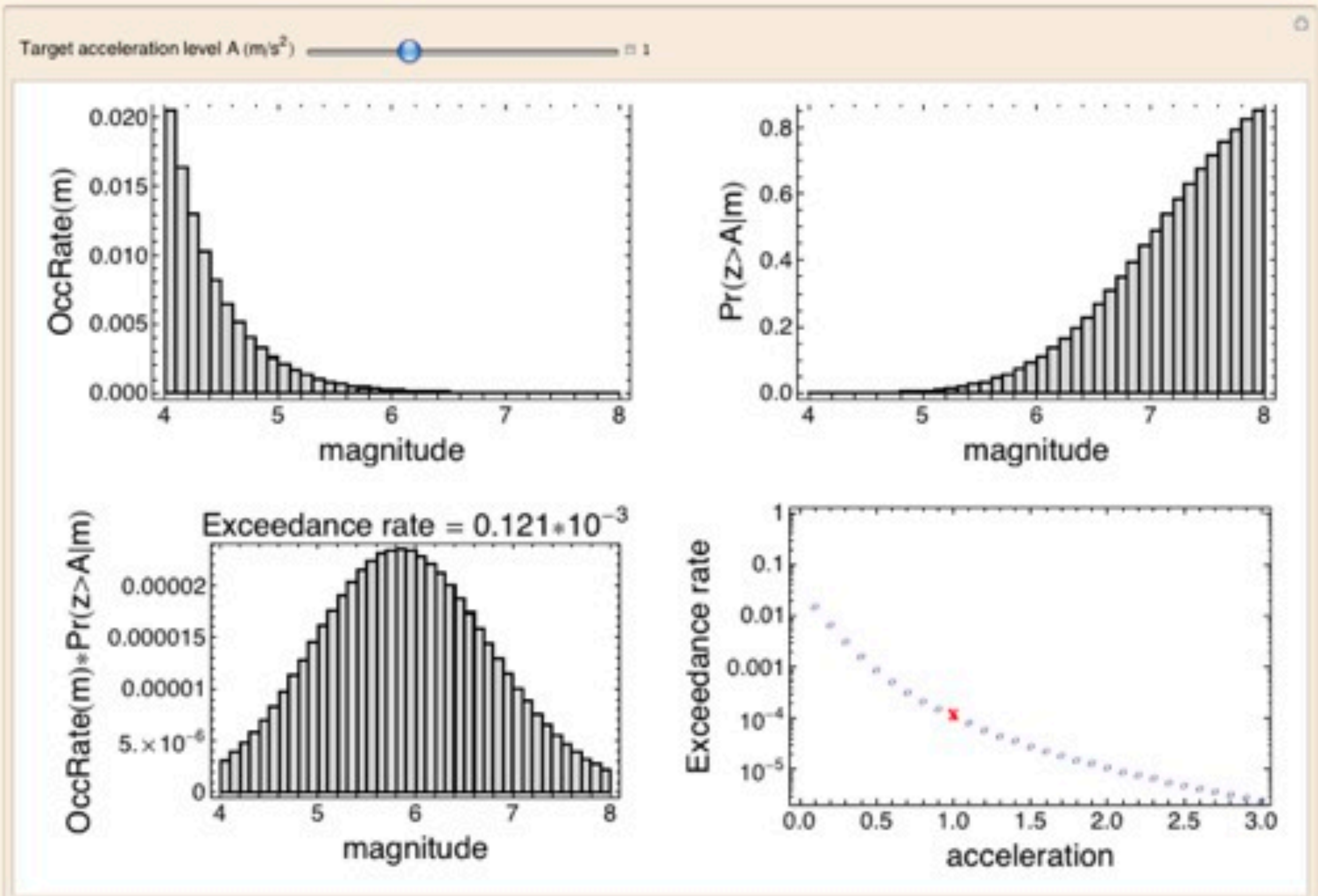
2) Multiply occurrence rate for each magnitude bin with
   $P(x>x_{test}$ | seismic event in magnitude bin)

   = expected exceedance rate for ground motion level of interest for each magnitude bin

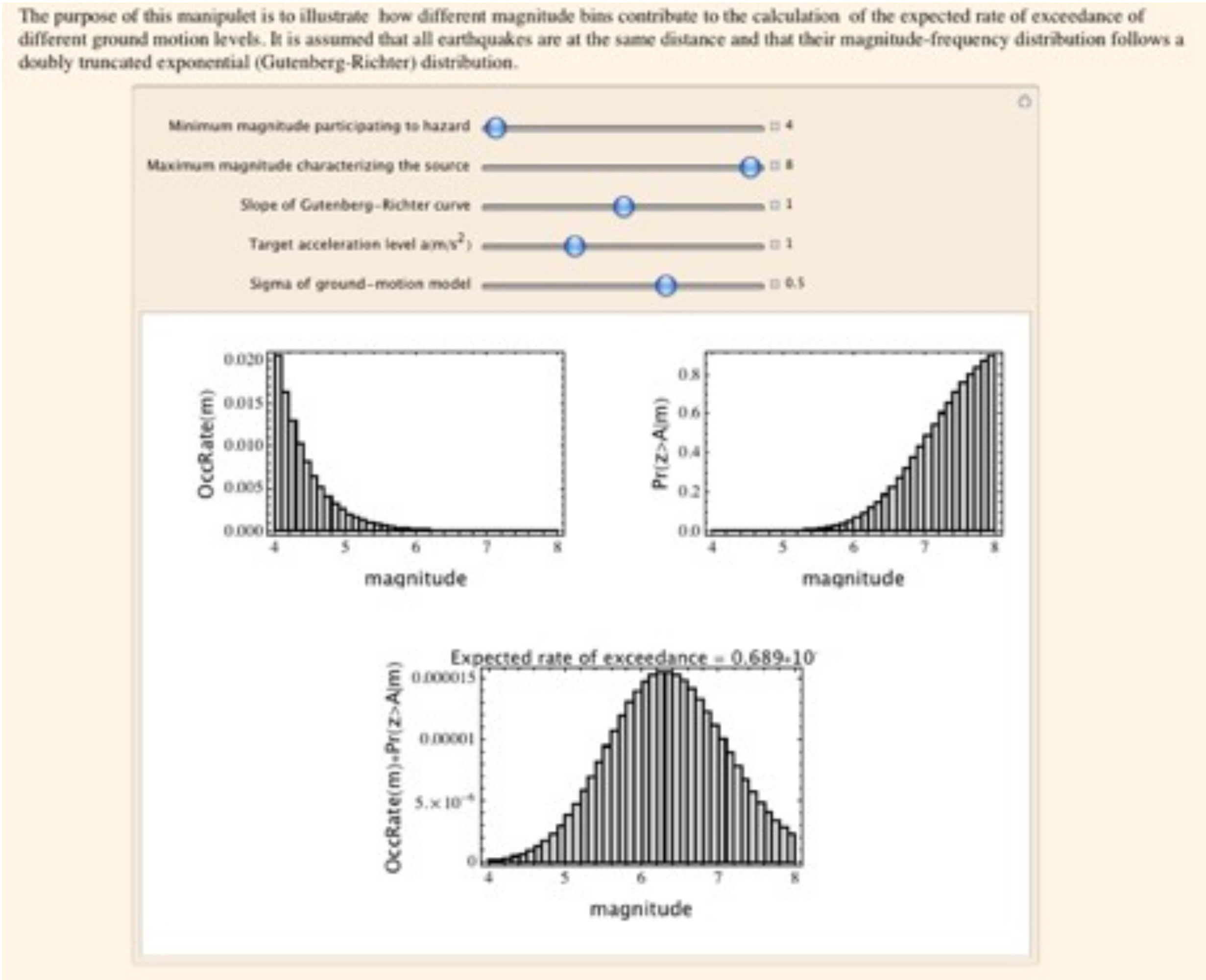
**Sum gives total expected exceedance rate for ground motion level of interest**

# ContribOfMagBins2HazardValue

The purpose of this manipulet is to illustrate how individual magnitude bins contribute to a the ground motion exceedance rate for a particular target acceleration value. It is assumed that all earthquakes are at the same distance and that their magnitude-frequency distribution follows a doubly truncated exponential (Gutenberg-Richter) distribution.

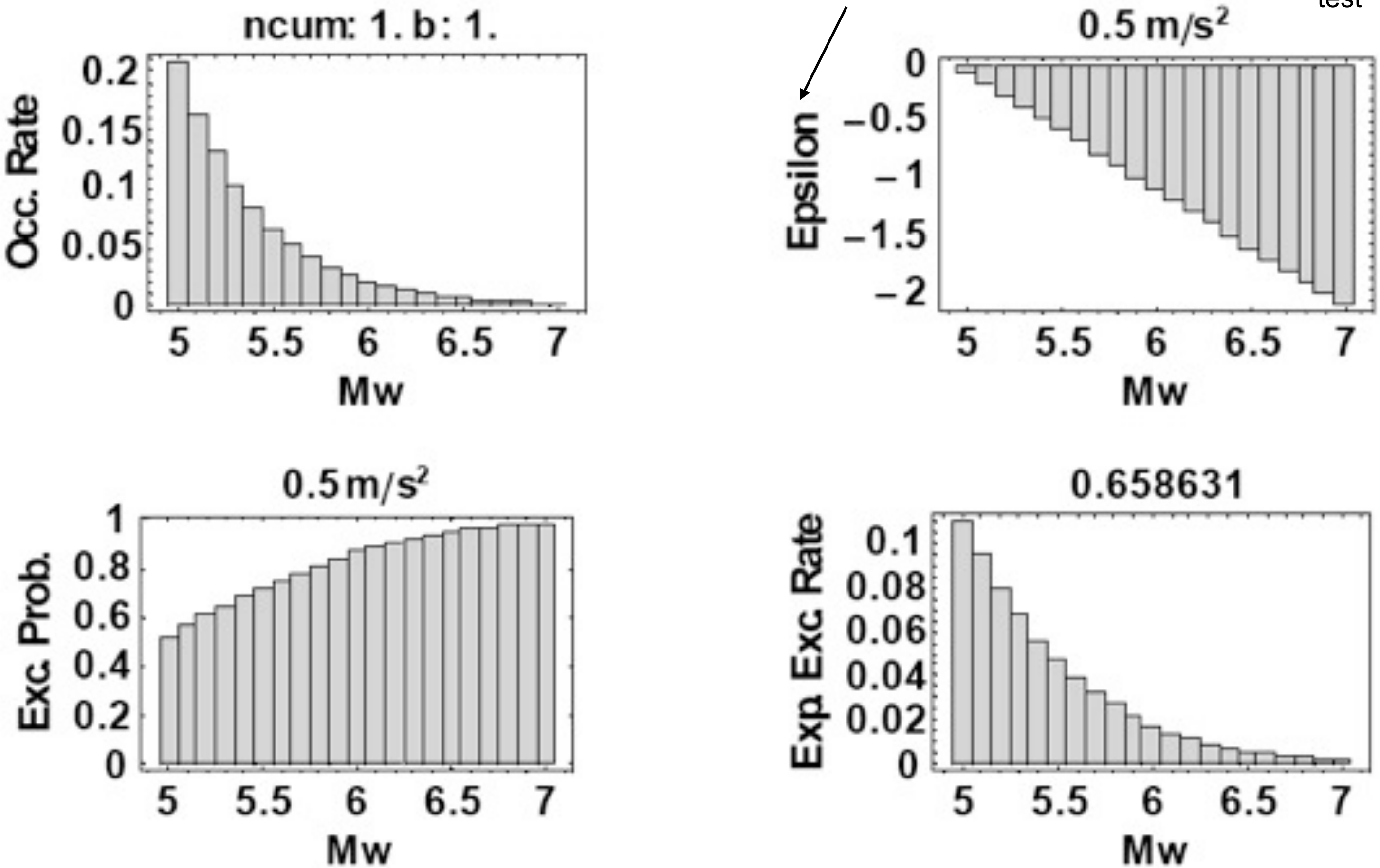


# HazFromDistribAtFixedDist



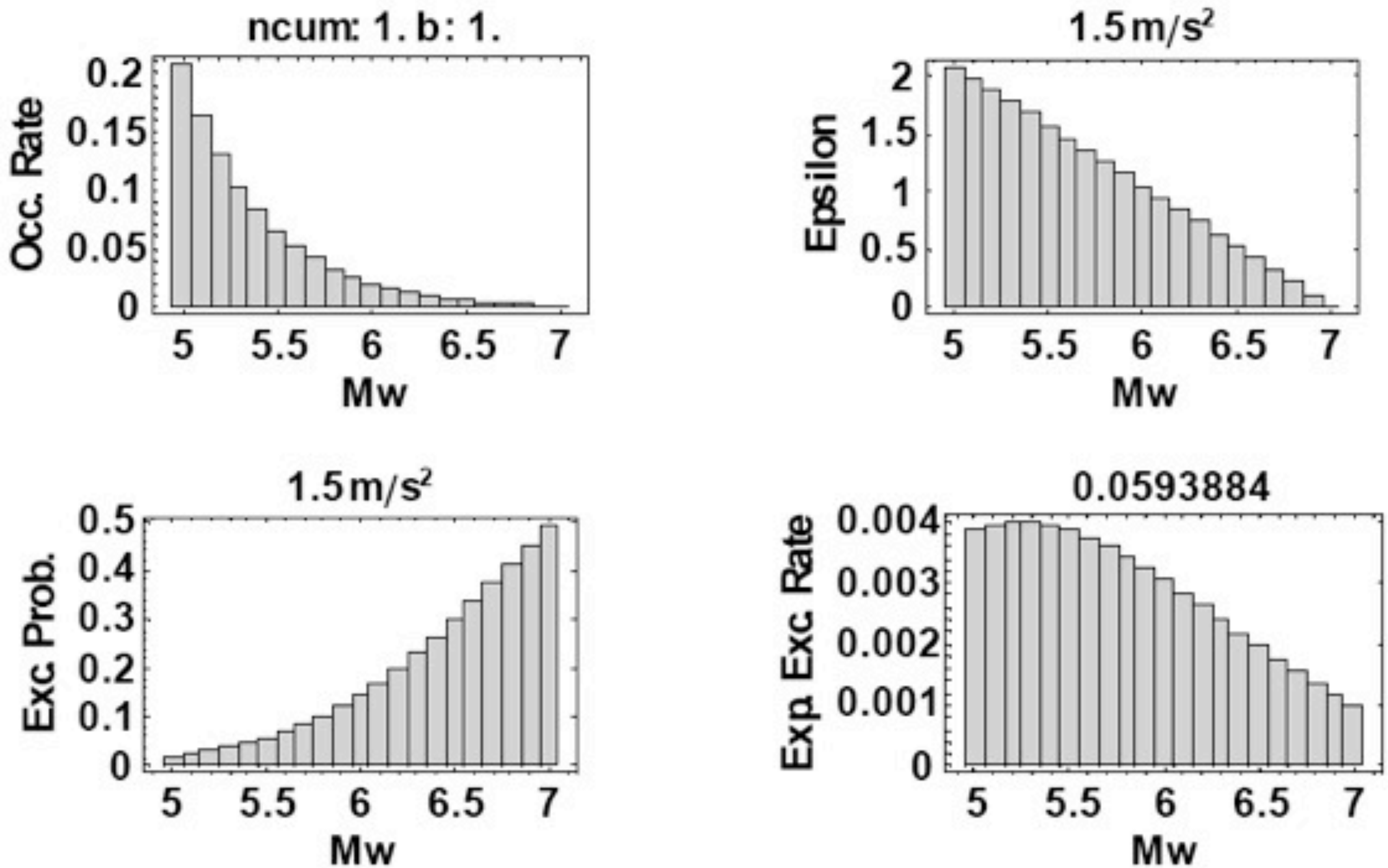
# $X_{test} = 0.5\ m/s^2$

# of standard deviation of $x_{test}$ from median



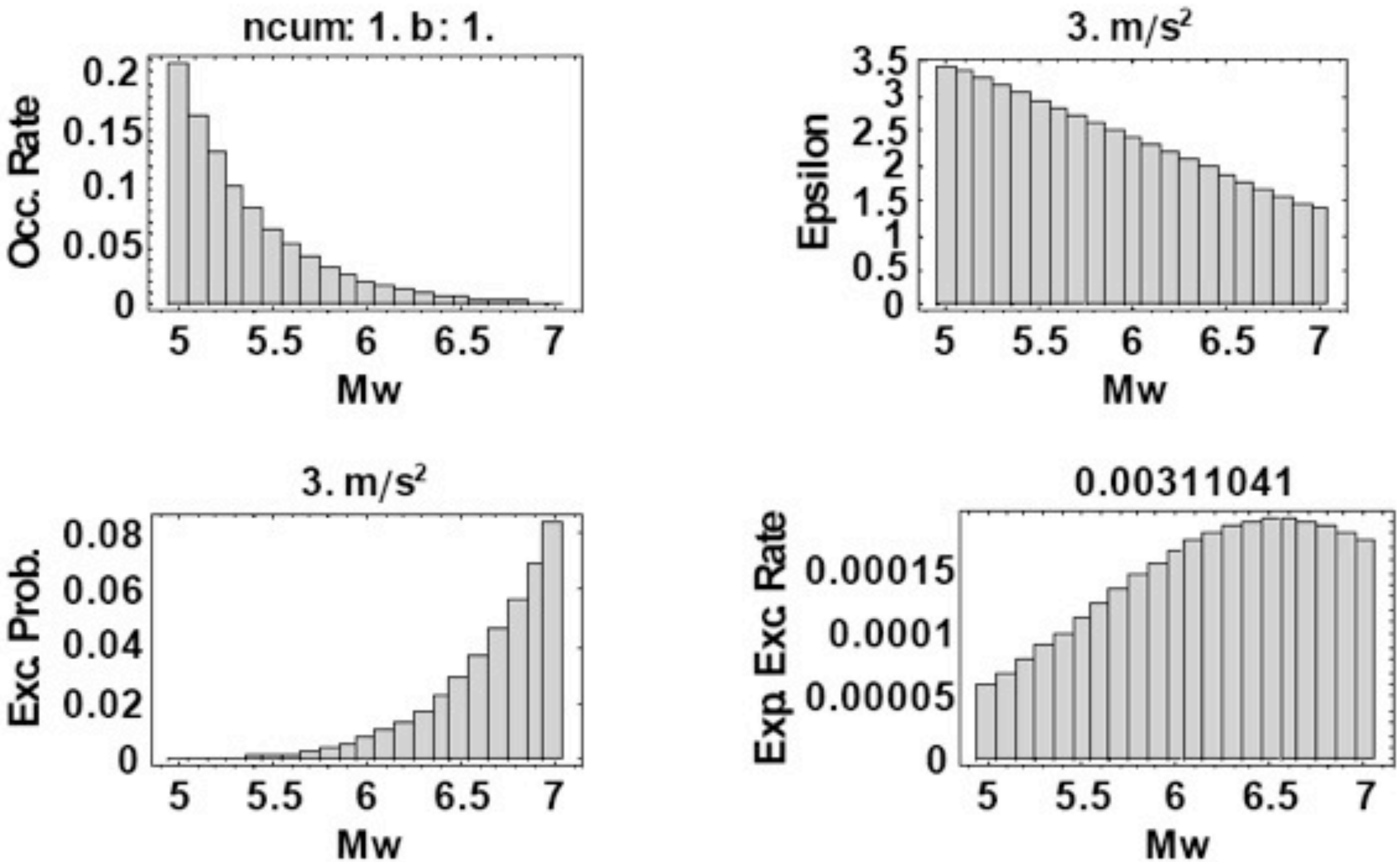

Expected exceedance rate for $0.5\ m/s^2$: 0.658631

# $X_{test}$= 1.5 m/s$^2$



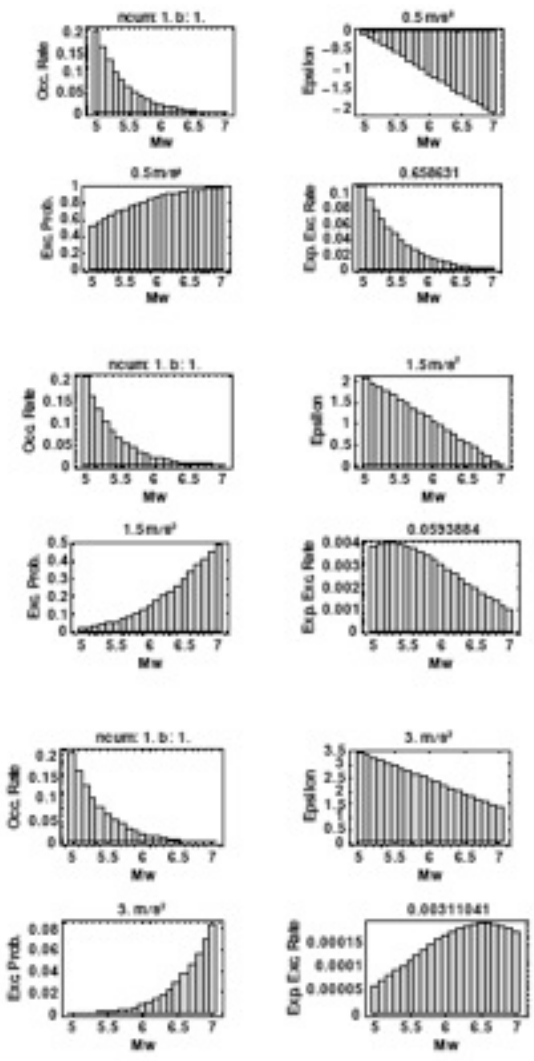

Expected exceedance rate for 1.5 m/s$^{2}$: 0.0593884

# $X_{test} = 3.0\ m/s^2$




Expected exceedance rate for 3.0 $m/s^{2}$: 0.004311041

# Lessons learned



• all sources contribute to the expected exceedance rate for a given ground-motion level of interest.

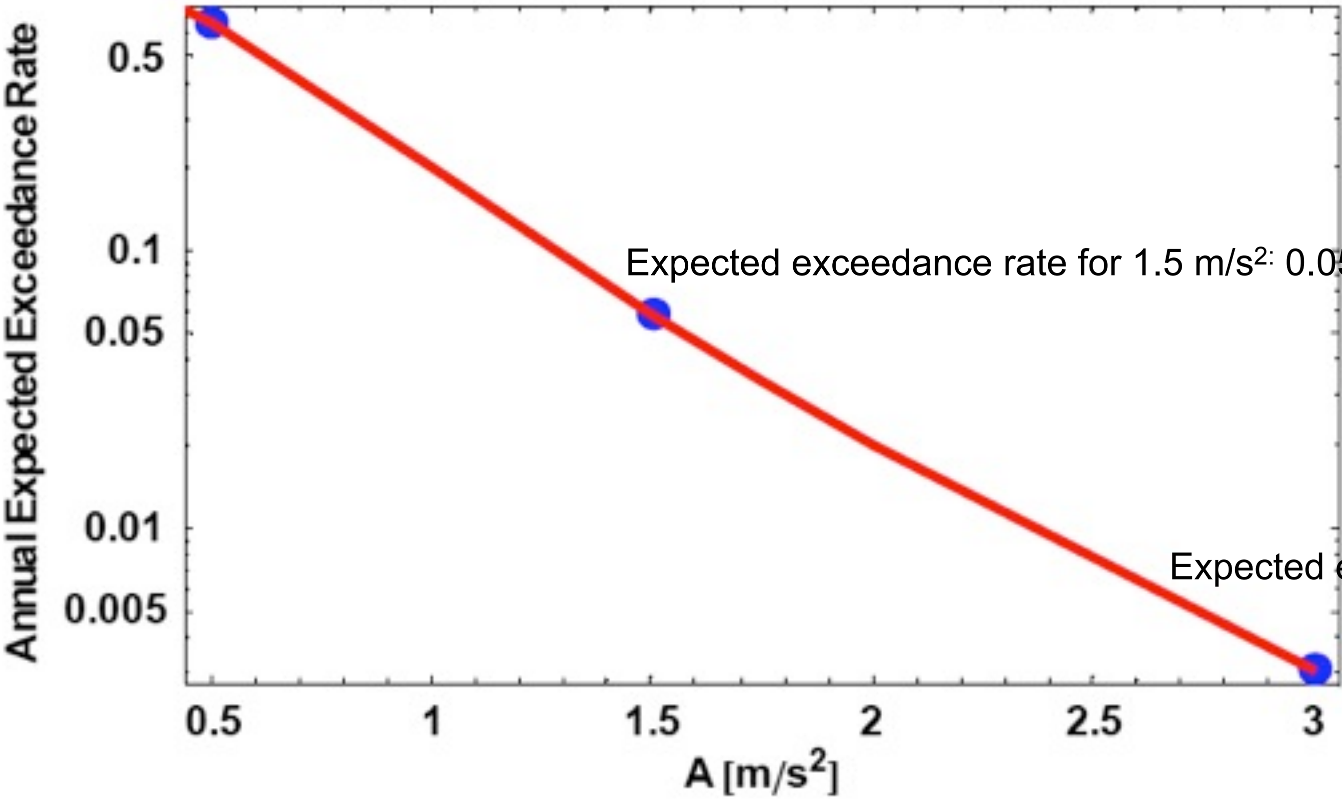
• the influence of the occurrence rates and the exceedance probabilities as a function of magnitude anti-correlate.

• the dominant contributions come from different magnitude bands depending on the ground motion level of interest. For higher ground-motion levels of interest, higher magnitude sources become more and more important.

# Expected Rate of Exceedance Curve

**Definition Hazard curve**:
Expected rate of exceedance as function of ground-motion level.

Expected exceedance rate for 0.5 m/s$^{2}$: 0.659




Expected exceedance rate for 1.5 m/s$^{2}$: 0.059

Expected exceedance rate for 3.0 m/s$^{2}$: 0.0043

# Definition

The reciprocal value of the expected rate of exceedance for a given ground-motion level is commonly referred to as **return period**.

# Return period and exceedance probability

**Assumption**: Poisson process with:

$\bar{N}$ = expected # of events in time $T_{idur}$

$\lambda = \bar{N} / T_{idur}$ = activity rate

$$P(\#\,of\ events = N\ \text{ in }\ T = T_{idur}) = \frac{\lambda^N}{N!} e^{-\lambda \cdot T_{idur}}$$

$$P(\#\,of\ events > 0\ \text{ in }\ T = T_{idur}) = 1 - \frac{\lambda^0}{0!} e^{-\lambda \cdot T_{idur}} = 1 - e^{-\lambda \cdot T_{idur}}$$

**Question:** Return period for ground motion produced as result of a Poissonian process in time for which the exceedance probability is 10% in T=50 years?

# Example: Poisson Process

$$P(\#\,of\ events > 0 \text{ in } T = T_{idur}) = 1 - \frac{\lambda^0}{0!} e^{-\lambda \cdot T_{idur}} \quad = 1 - e^{-\lambda \cdot T_{idur}}$$

$$\Rightarrow \mathrm{P}(\#\text{ of events} > 0 \text{ in } \mathrm{T} = 50\mathrm{a}) = 0.1 \;= 1 - \mathrm{e}^{-\lambda \cdot 50}$$

$$\Rightarrow \lambda = \frac{\ln(1-0.1)}{50} \quad = 0.00211$$

**Remember:** Return period is 1/(annual exceedance rate l)

$$\Rightarrow \text{Return period} = 1/0.00211 = 474\, years$$

# Potential misunderstandings

The term return period has led to a number of misunderstandings.

• it **wrongly suggests** some periodicity in the process of generating ground motion and

• it might **wrongly** create the impression that ground motion corresponding to large return periods is caused by earthquakes which have a large average inter-event time.

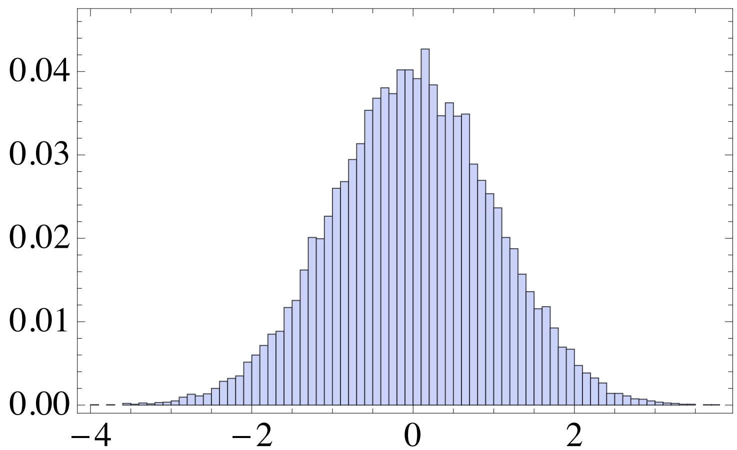
# Memory test

*How many random numbers do you need to generate* ***on average*** *to obtain a number larger than 1 σ, 2 σ, 3 σ, 4 σ?*

# Answer from simulation

$N(0,1)$ 20000 samples

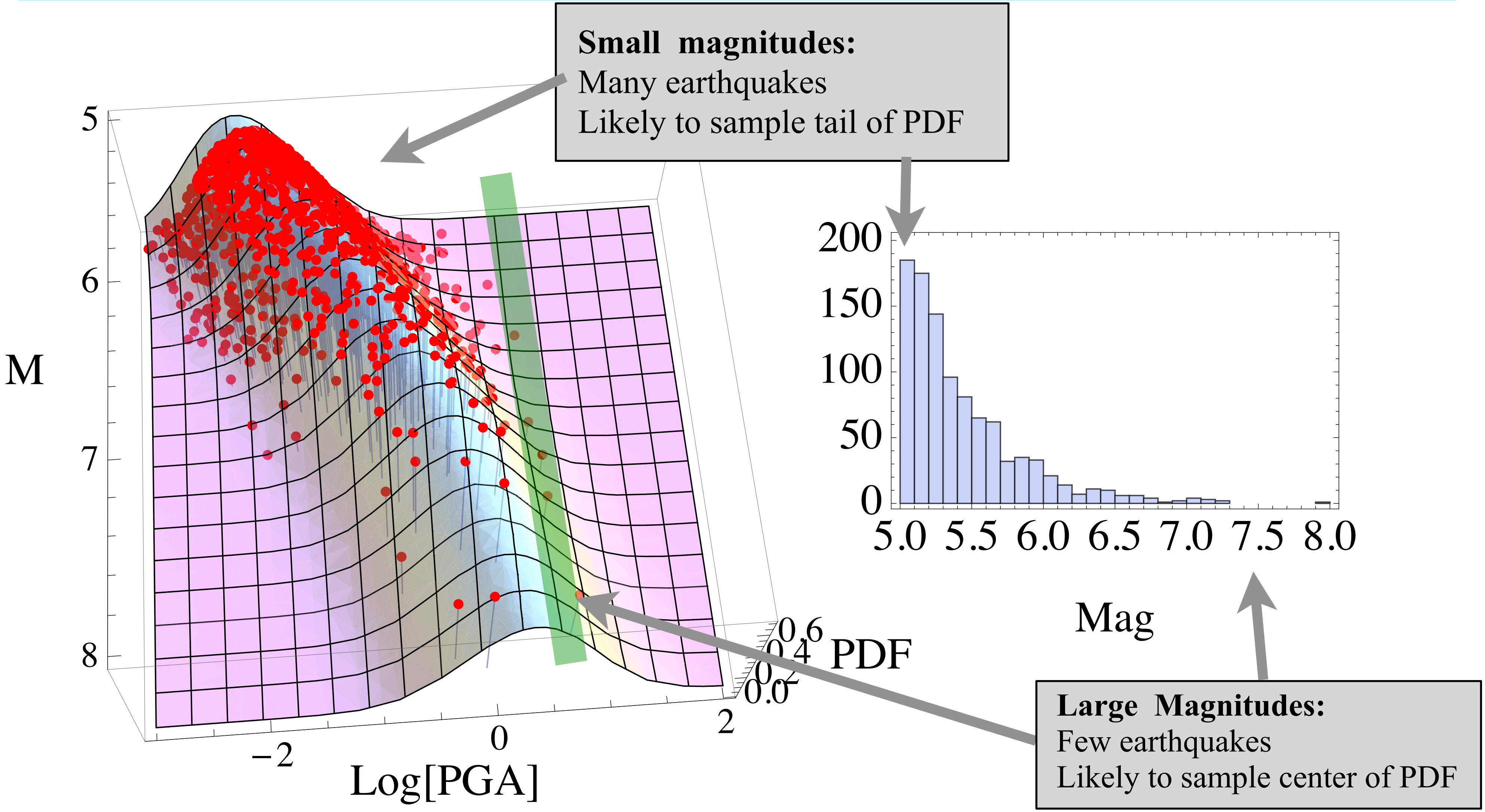
How many trials to reach 1σ



| 1 σ | 2 σ | 3 σ | 4 σ |
|---|---|---|---|
| 2.38 | 17.64 | 399.28 | 14182.3 |

For a small number of trials, it is very likely that random numbers generated from a normal distribution stay close to the center of the distribution.

# Consequences for seismic ground motion



**Ground motion corresponding to large return periods can either be caused by typical ground motion from rare events or by untypical ground motion from frequent events.**

From the center of the probability distribution

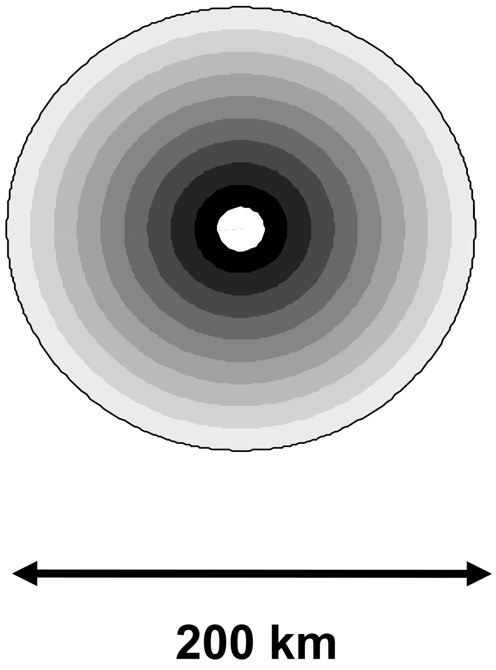
From the tails of the probability distribution

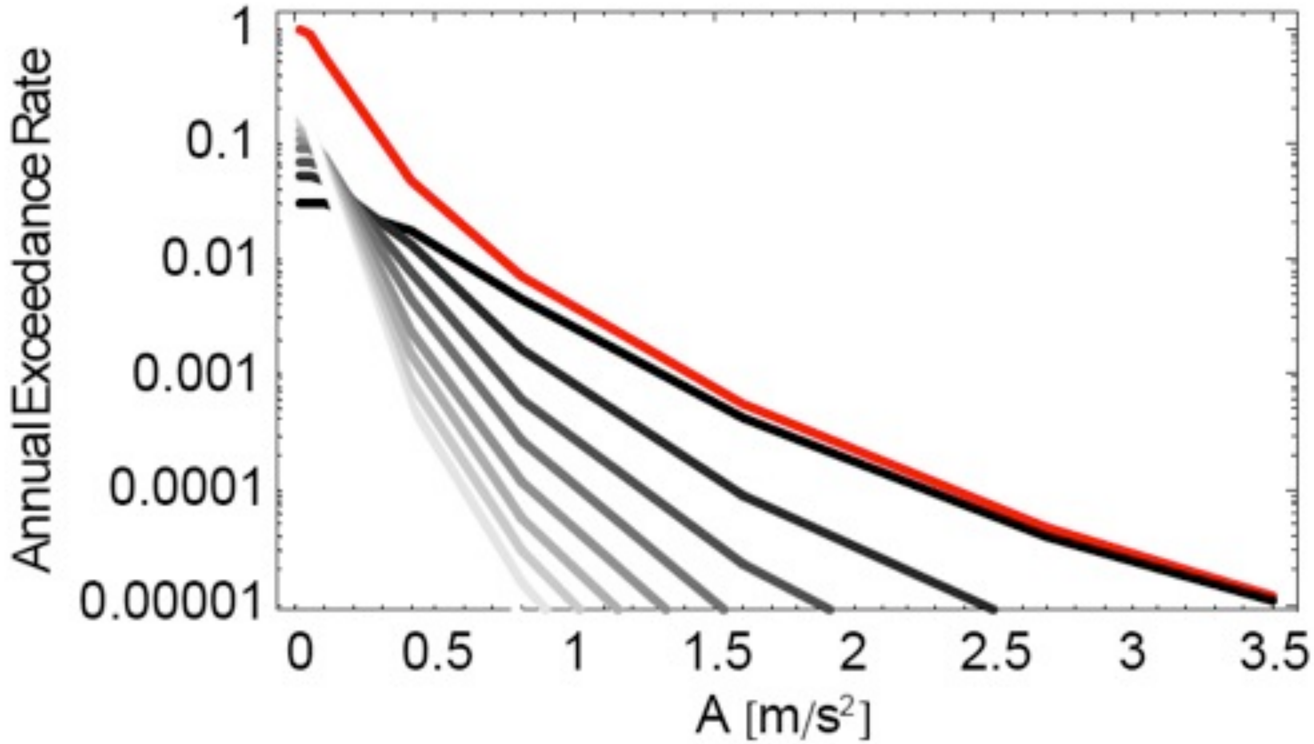
**...It´s not just the occurrence rate of earthquakes!**

# Earthquake distribution from within a circular area


Frank Scherbaum

# Uniformly distributed sources within a circular area




$[M_{low}, M_{upp}] = [4,7]$

Total # of earthquake with M > 4: 1 per year

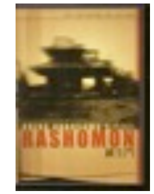
b-value: 1

No earthquakes in distances less than 10 km
(out of curiosity)

**Different hazard levels are dominated by sources in different distances.**

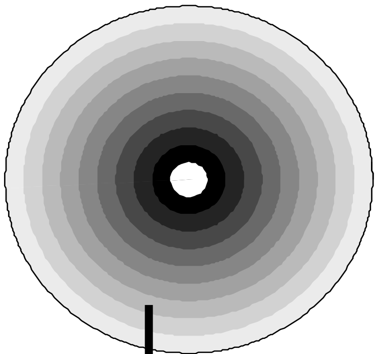


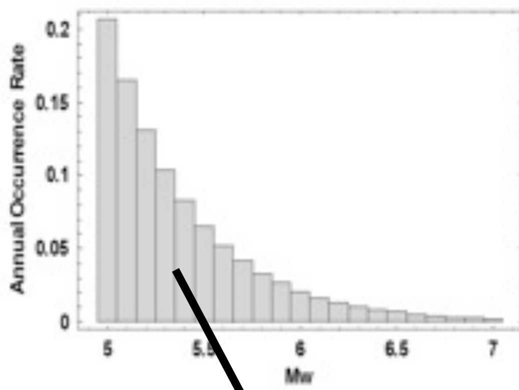
Frank Scherbaum




# Theoretical perspective

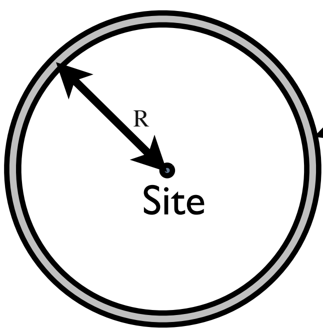
# Contribution to hazard from one distance and one magnitude bin






Fixed distance (one ring)

Fixed magnitude range (one bin)

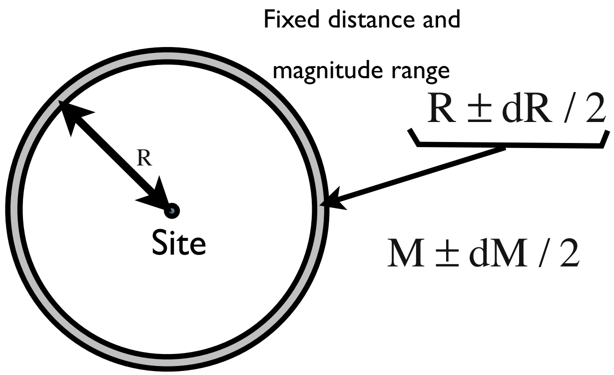



$R \pm dR / 2$ $M \pm dM / 2$

Hazard contribution of single magnitude-distance-bin:

$$P(R \pm dR / 2, M \pm dM / 2, Sa > a)$$

Joint probability that an earthquake within a magnitude range $M \pm dM/2$ occurs within a distance range of $R \pm dR/2$ and causes the ground motion level *a* to be exceeded (Sa > *a*)




$$P(R \pm dR/2, M \pm dM/2, Sa > a)$$

Rewrite joint probability:

$$P(R \pm dR/2, M \pm dM/2, Sa > a) = P(Sa > a \mid R \pm dR/2, M \pm dM/2) \cdot P(R \pm dR/2, M \pm dM/2)$$

conditional probability for ground motion exceedance

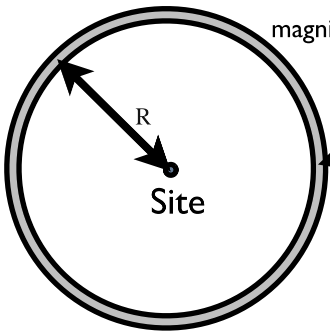
joint probability for earthquake occurrence in a magnitude-distance-bin

Assuming independence of M and R:

$$= P(Sa > a \mid R \pm dR/2, M \pm dM/2) \cdot P(R \pm dR/2) \cdot P(M \pm dM/2)$$

Fixed distance and magnitude range

$R \pm dR/2$

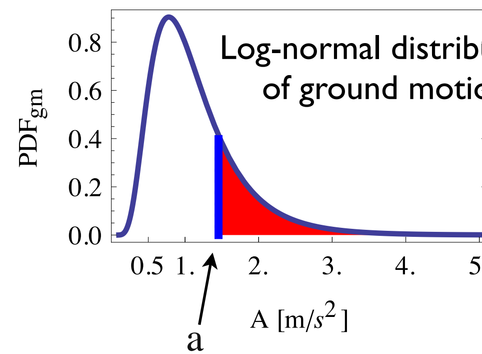



$M \pm dM/2$

$$P(R \pm dR/2, M \pm dM/2, Sa > a) =$$
$$P(Sa > a \mid R \pm dR/2, M \pm dM/2) \cdot P(R \pm dR/2) \cdot P(M \pm dM/2)$$

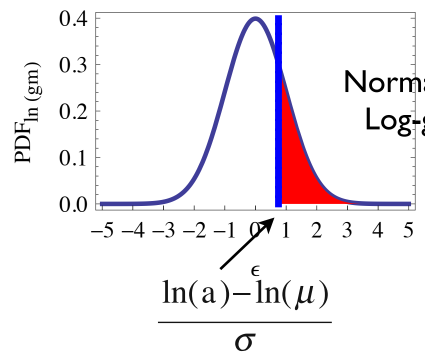
Evaluate exceedance probability:

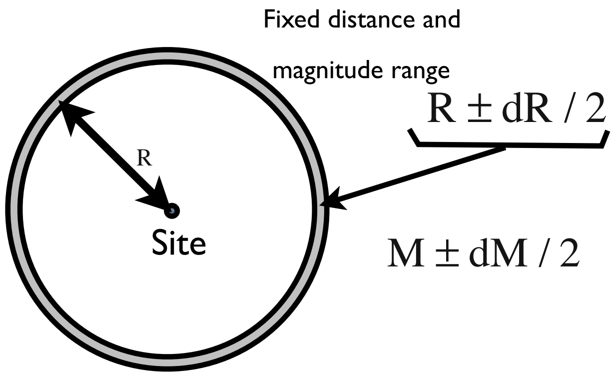
$$\int_a^{\infty} PDF_{gm}(y \mid R \pm dR/2, M \pm dM/2) \cdot dy$$




Considering natural logarithm of ground motion instead:

$$\int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \underbrace{PDF_{\ln(gm)}(\varepsilon \mid R \pm dR/2, M \pm dM/2)}_{\text{Ground motion model}} \cdot d\varepsilon$$






Assumption: Distance distribution $f_r(R)$ within source zone

$$P(R \pm dR / 2) = f_R(R) \cdot dR$$

Assumption: Magnitude distribution $f_m(M)$

$$P(M \pm dM / 2) = f_m(M) \cdot dM$$

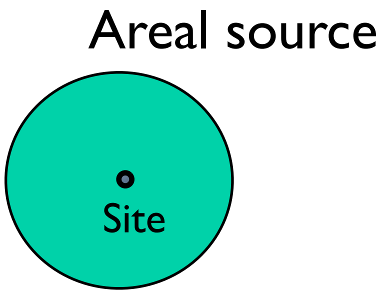
## Hazard contribution:

$$P(R \pm dR / 2, M \pm dM / 2, Sa > a)$$

$$= P(Sa > a | R \pm dR / 2, M \pm dM / 2) \cdot P(R \pm dR / 2) \cdot P(M \pm dM / 2)$$

$$= P(Sa > a | R, M) \cdot f_R(R) \cdot f_m(M) \cdot dR \cdot dM$$

# Spatial distribution of earthquakes




Exceedance probability:

$$P(Sa > a) = \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} P(Sa > a \mid M,R) \cdot f_R(R) \cdot f_m(M) \cdot dR \cdot dM$$

Expected exceedance rate: $\nu(Sa > a) = P(Sa > a) \cdot N(M_{min})$

$$\nu(Sa > a) = N(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} P(Sa > a \mid M,R) \cdot f_R(R) \cdot f_m(M) \cdot dR \cdot dM$$

# Several areal sources

Single source:

$$\nu(Sa > a) = N(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} P(Sa > a \mid M,R) \cdot f_R(R) \cdot f_m(M) \cdot dR \cdot dM$$

Several sources:

$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \underbrace{P(Sa > a \mid M,R)}_{\text{explictely expressed as integral over } \varepsilon} \cdot f_R(R) \cdot f_m(M) \cdot dR \cdot dM$$

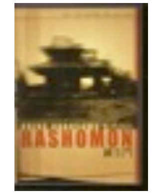
explictely expressed as

integral over ε

Different notation (Abrahamson, 2000):

$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \overbrace{\int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} PDF_{\ln(gm)}(\varepsilon \mid R,M)} \cdot f_R(R) \cdot f_m(M) \cdot d\varepsilon \cdot dR \cdot dM$$

**Hazard Integral**




# The hazard integral illustrated

# What´s in the hazard integral?

$$\nu(\mathrm{Sa} > \mathrm{a}) = \sum_{i=1}^{n_{sources}} \mathrm{N_i}(\mathrm{M_{min}}) \cdot \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} \int_{\mathrm{R_{min}}}^{\mathrm{R_{max}}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \mathrm{PDF_{\ln(gm)}}(\varepsilon|\mathrm{R},\mathrm{M}) \cdot \mathrm{f_R}(\mathrm{R}) \cdot \mathrm{f_m}(\mathrm{M}) \cdot d\varepsilon \cdot d\mathrm{R} \cdot d\mathrm{M}$$

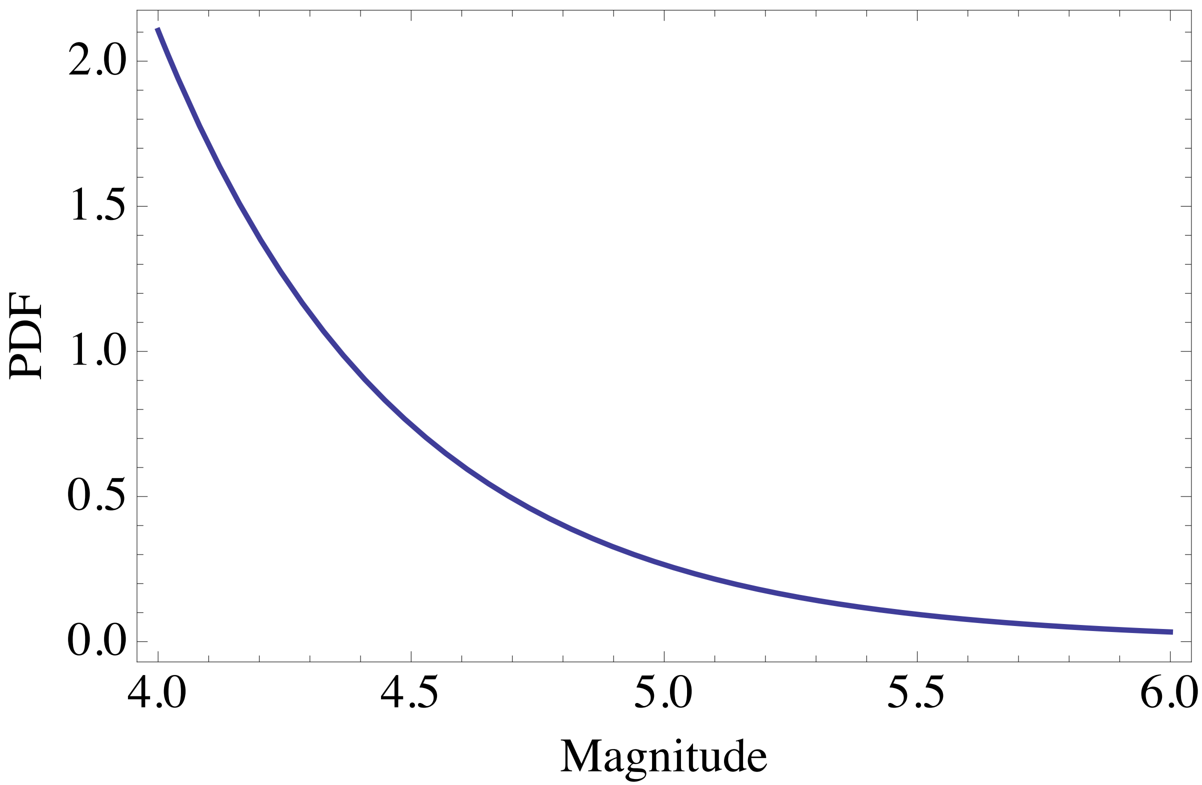
Ground motion model

Distance distribution

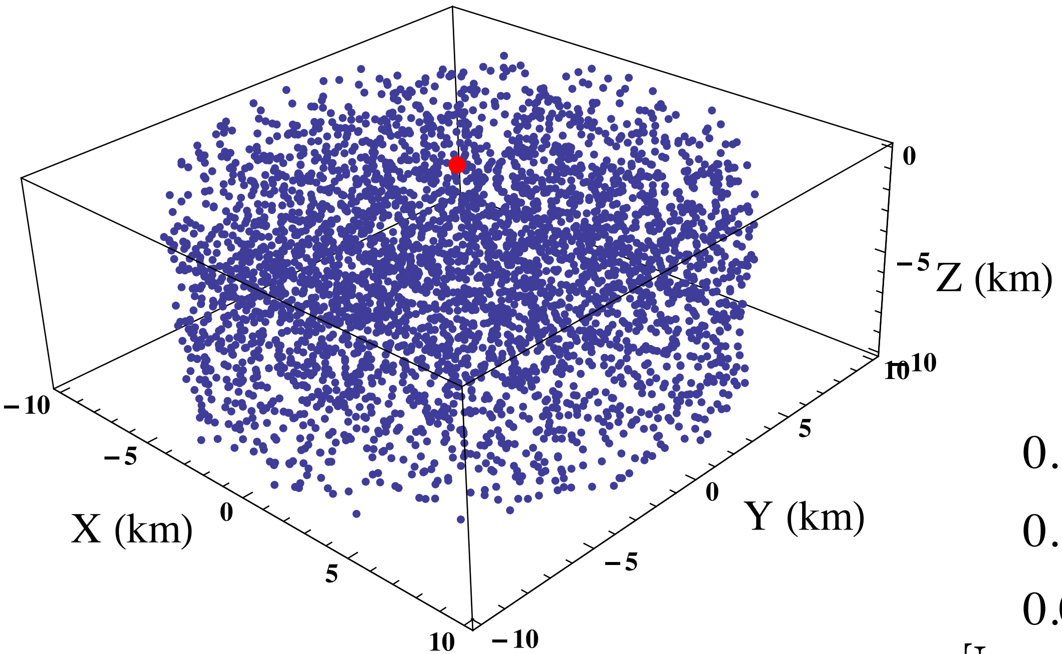
Magnitude distribution

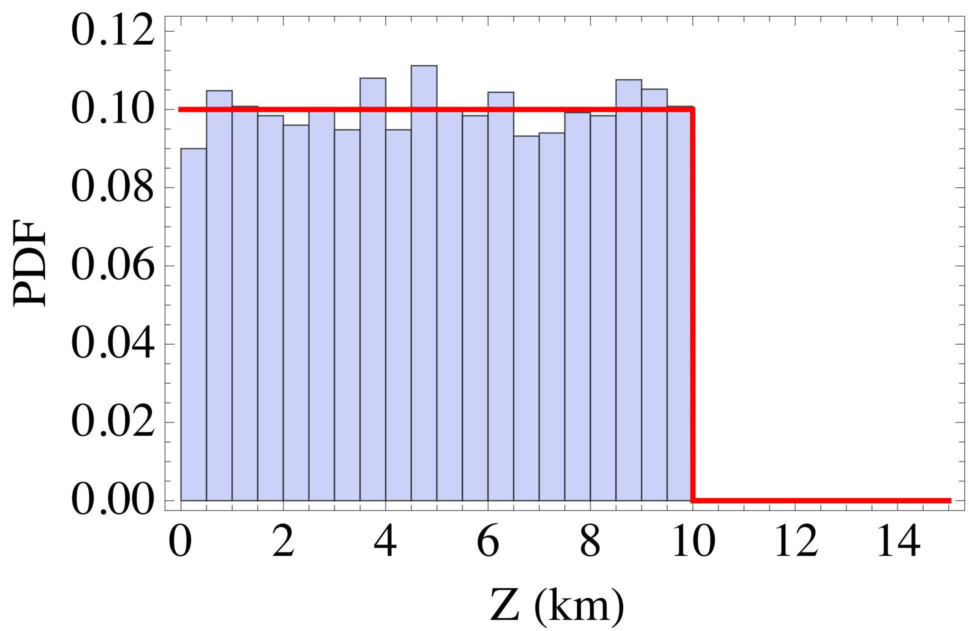
# The magnitude distribution

Doubly truncated exponential distribution
(Gutenberg-Richter)

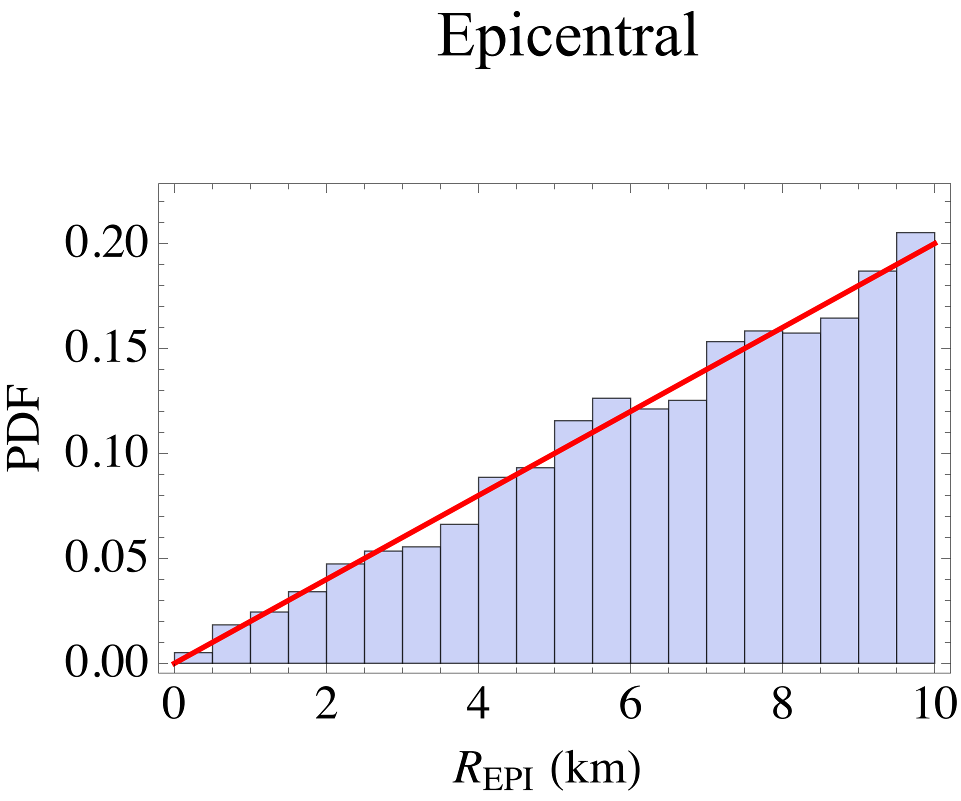


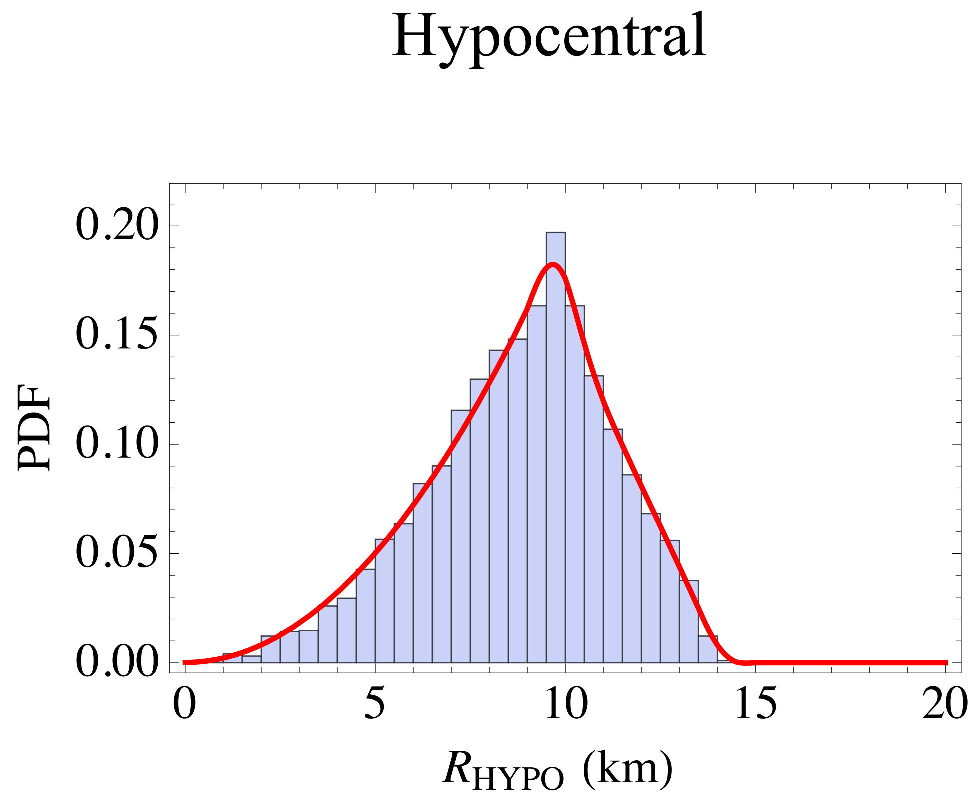
# Source distribution
# unrealistic but simple



# Distance distribution



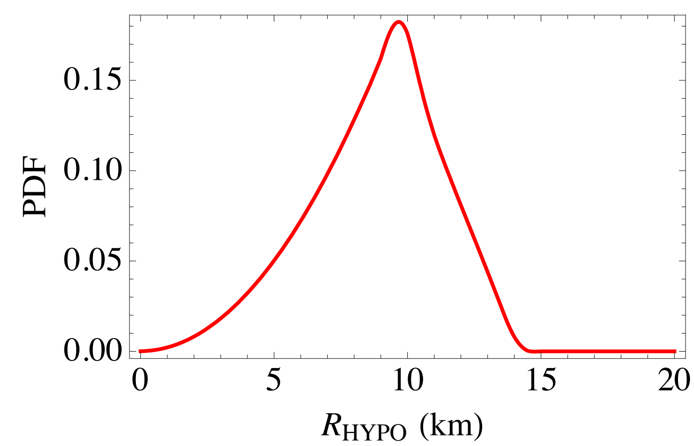



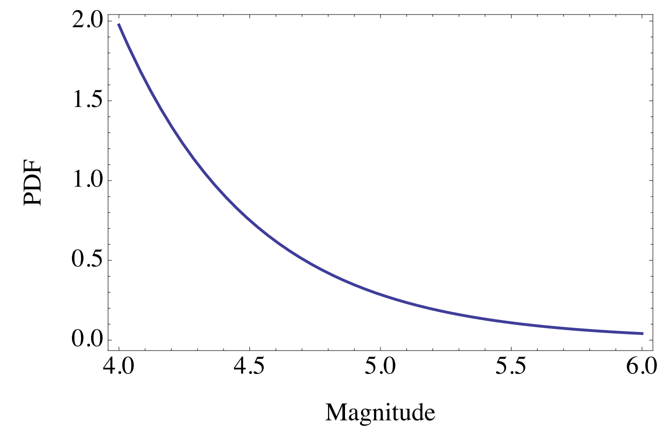
# What´s in the hazard integral?

$$\nu(\mathrm{Sa} > \mathrm{a}) = \sum_{i=1}^{n_{\mathrm{sources}}} \mathrm{N_i}(\mathrm{M_{min}}) \cdot \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} \int_{\mathrm{R_{min}}}^{\mathrm{R_{max}}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \mathrm{PDF}_{\ln(gm)}(\varepsilon | \mathrm{R},\mathrm{M}) \cdot \mathrm{f_R}(\mathrm{R}) \cdot \mathrm{f_m}(\mathrm{M}) \cdot d\varepsilon \cdot d\mathrm{R} \cdot d\mathrm{M}$$

Distance distribution

Magnitude distribution



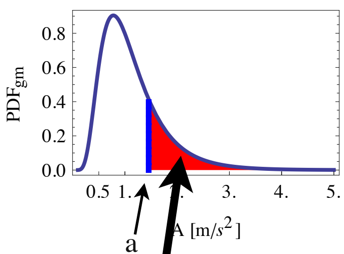



# What´s in the hazard integral?

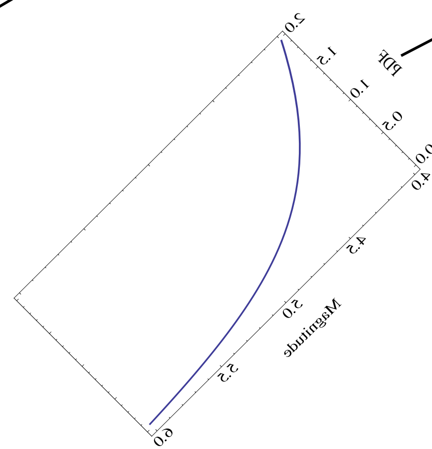
$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} PDF_{\ln(gm)}(\varepsilon | R, M) \cdot f_R(R) \cdot f_m(M) \cdot d\varepsilon \cdot dR \cdot dM$$

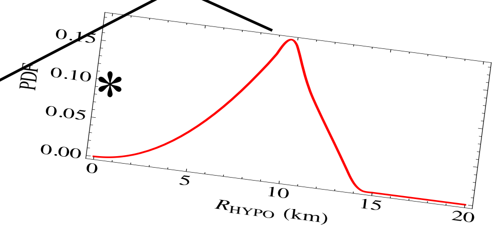
Ground motion model

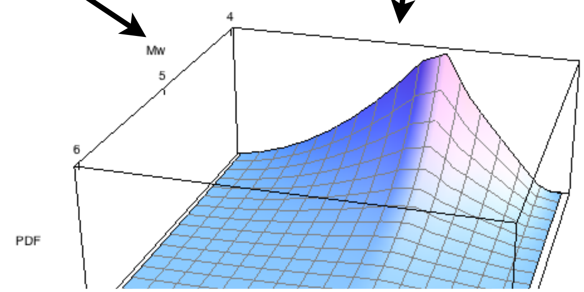
Distribution of ground motion Log-normal




Probability of exceedance of $a$ is function of magnitude and distance



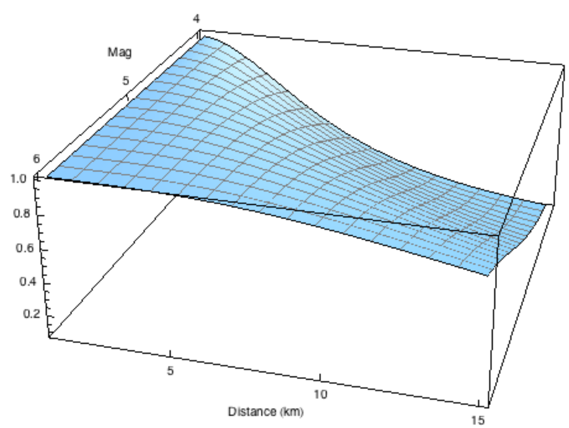


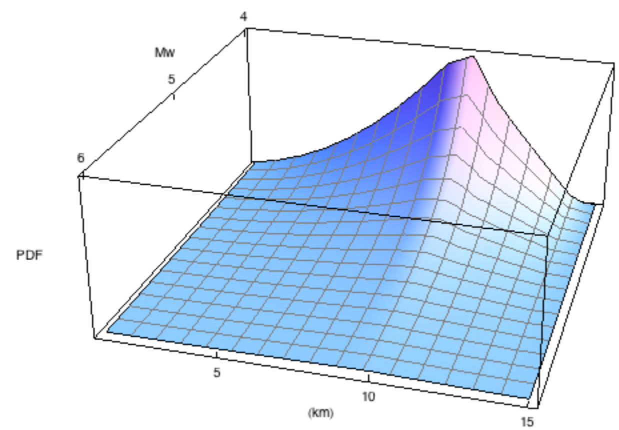
GMMProbExceedance

# What´s in the hazard integral?

$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \underbrace{PDF_{\ln(gm)}(\varepsilon | R,M)} \cdot \underbrace{f_R(R) \cdot f_m(M)} \cdot d\varepsilon \cdot dR \cdot dM$$

1 m/s$^2$



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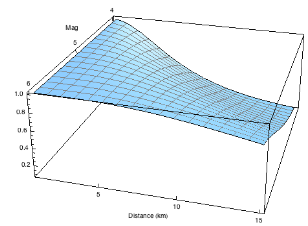
# What´s in the hazard integral?

$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \underbrace{PDF_{\ln(gm)}(\varepsilon|R,M)} \cdot \underbrace{f_R(R) \cdot f_m(M)} \cdot d\varepsilon \cdot dR \cdot dM$$

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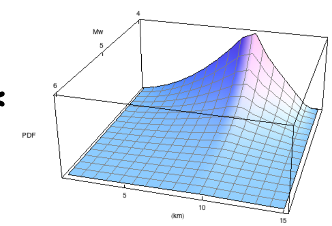
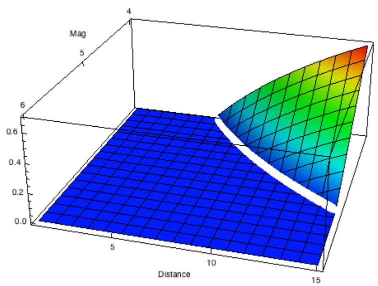
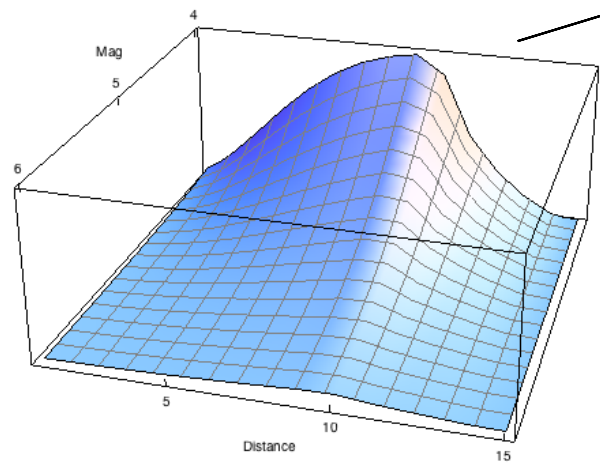



Corresponding
distribution
of ε

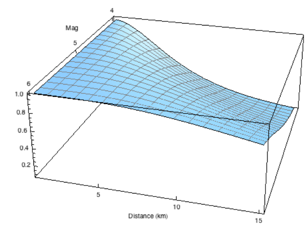



use ε for shading



# What´s in the hazard integral?

$$\nu(\mathrm{Sa} > \mathrm{a}) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \underbrace{PDF_{\ln(gm)}(\varepsilon | R,M)} \cdot \underbrace{f_R(R) \cdot f_m(M)} \cdot d\varepsilon \cdot dR \cdot dM$$



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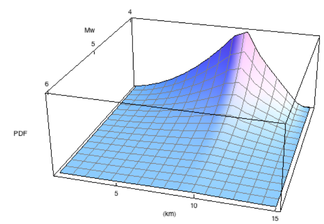
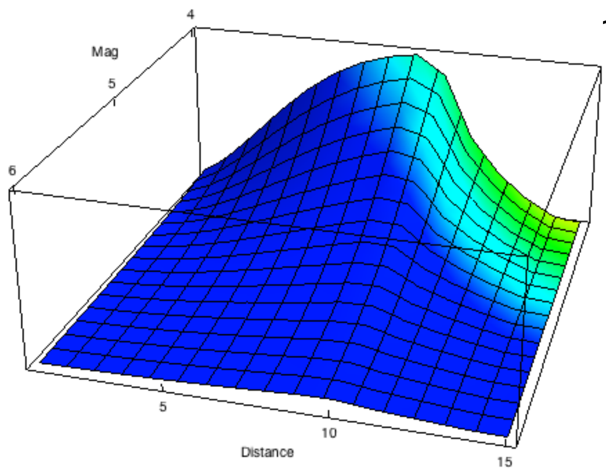
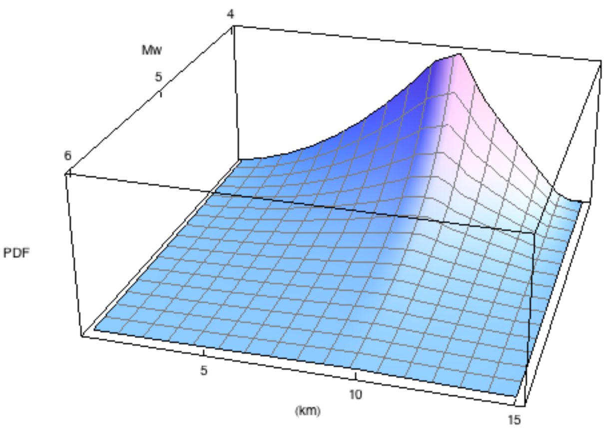


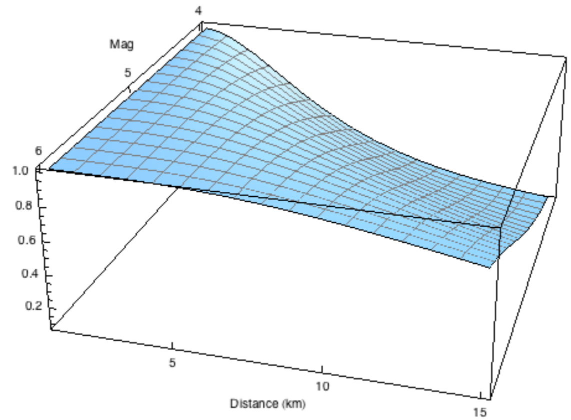



The value of the integral is simply the volume under the surface

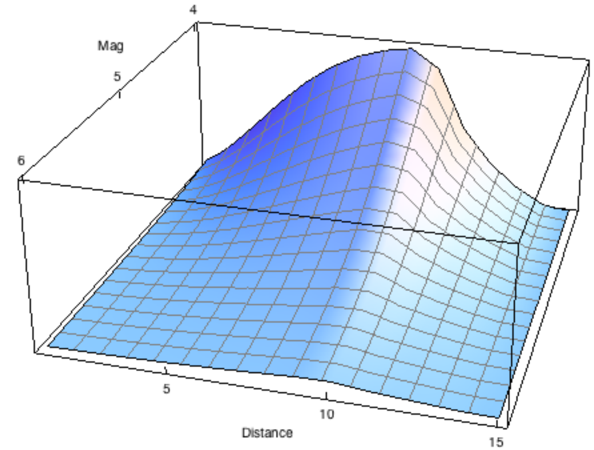
# 1 m/s$^2$



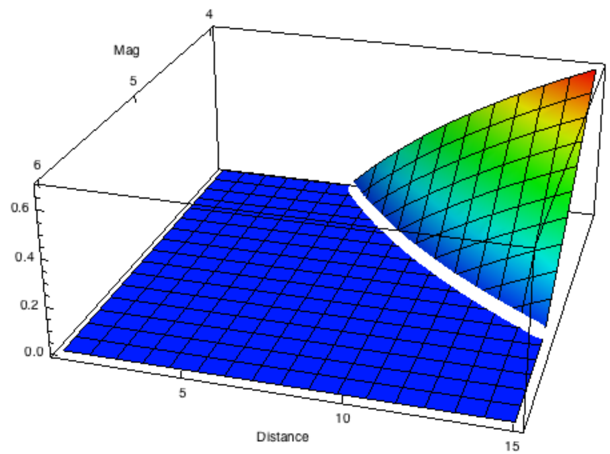
Mw
4
5
6
PDF
5
10
15
(km)

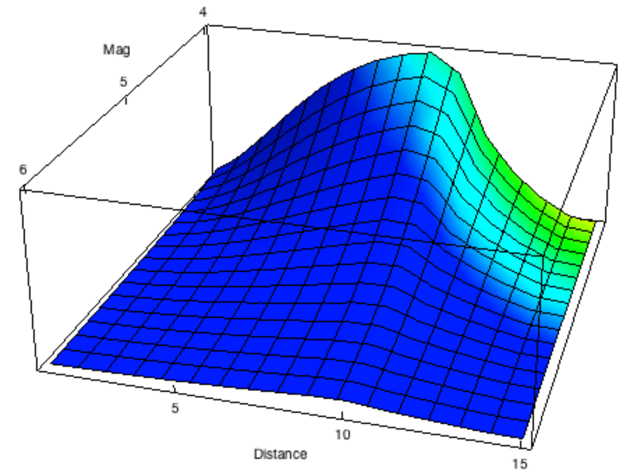
Mag
4
5
6
1.0
0.8
0.6
0.4
0.2
5
10
15
Distance (km)

Mag
4
5
6
5
10
15
Distance

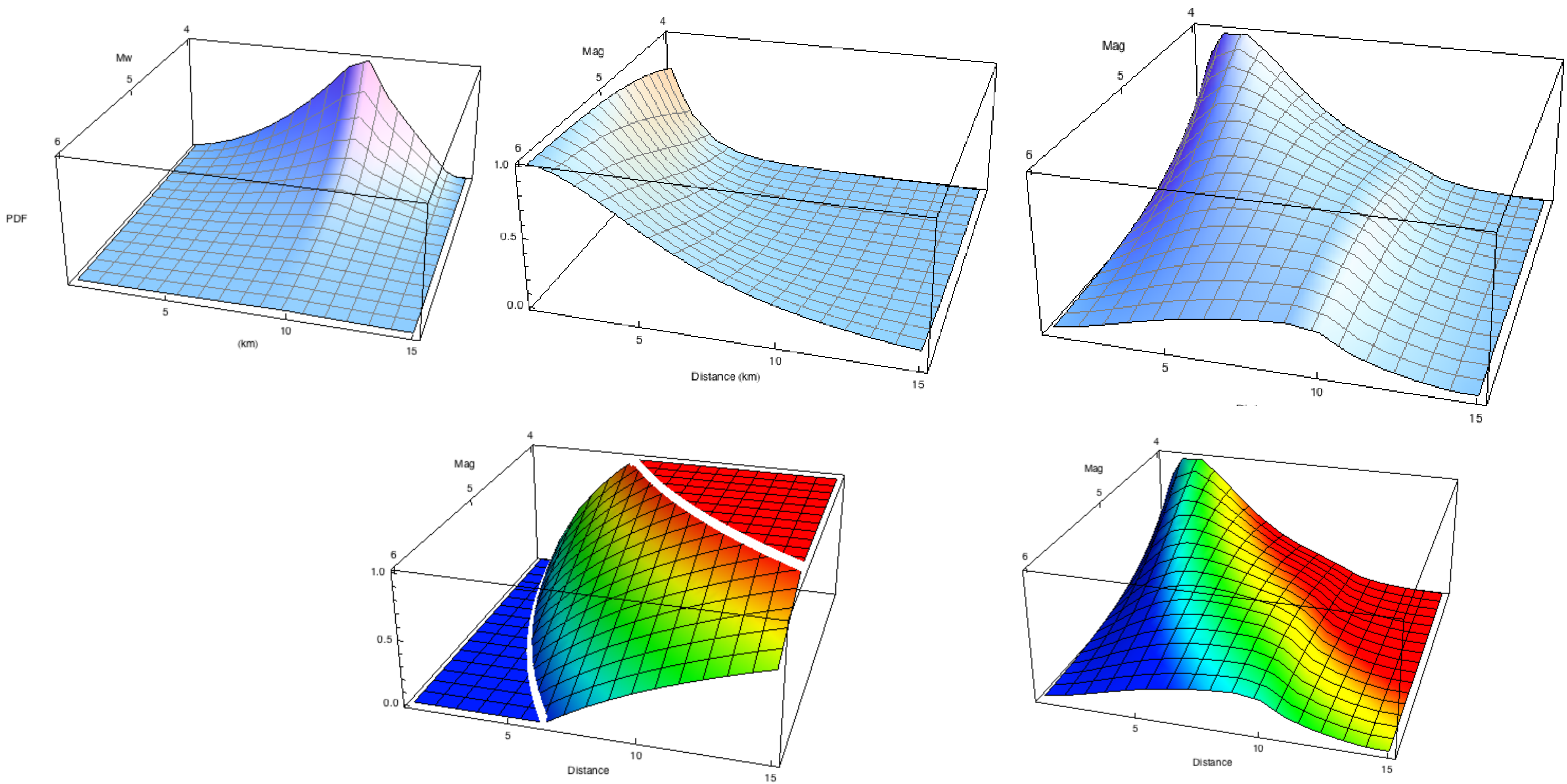



Mag
4
5
6
0.6
0.4
0.2
0.0
5
10
15
Distance

Mag
4
5
6
5
10
15
Distance

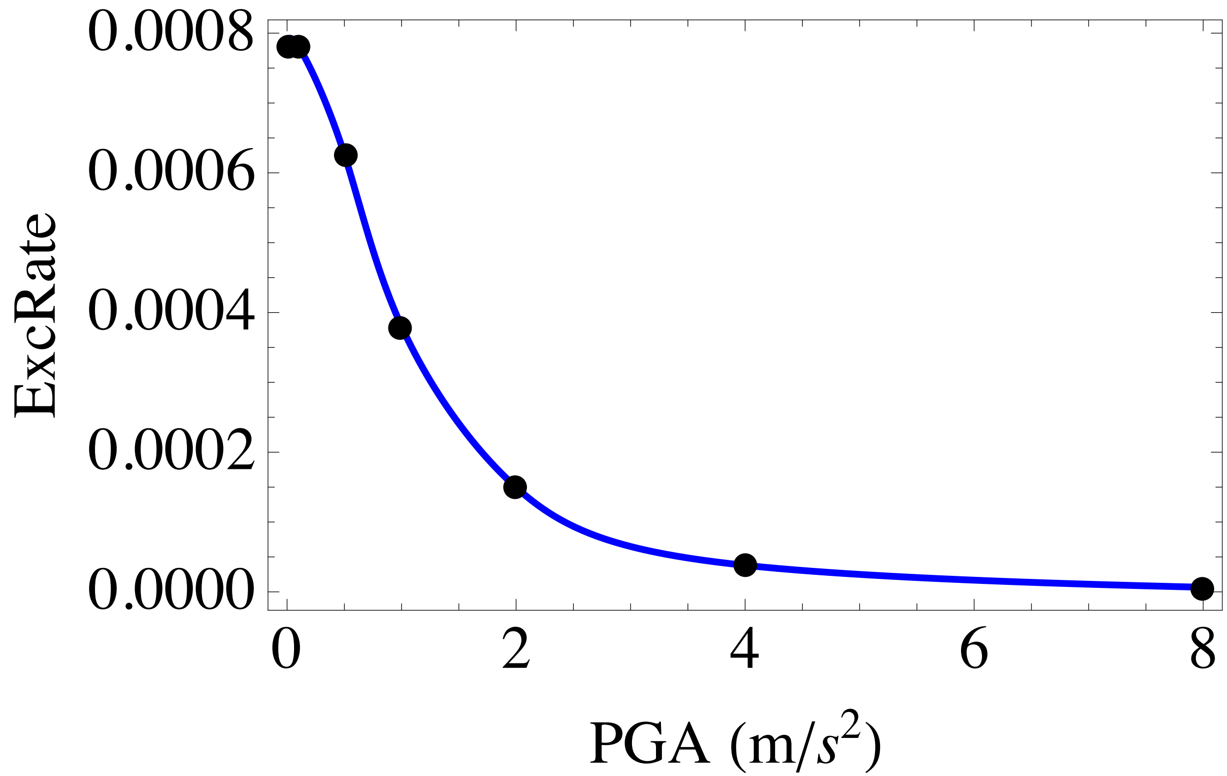
# 4 m/s$^2$






Frank Scherbaum

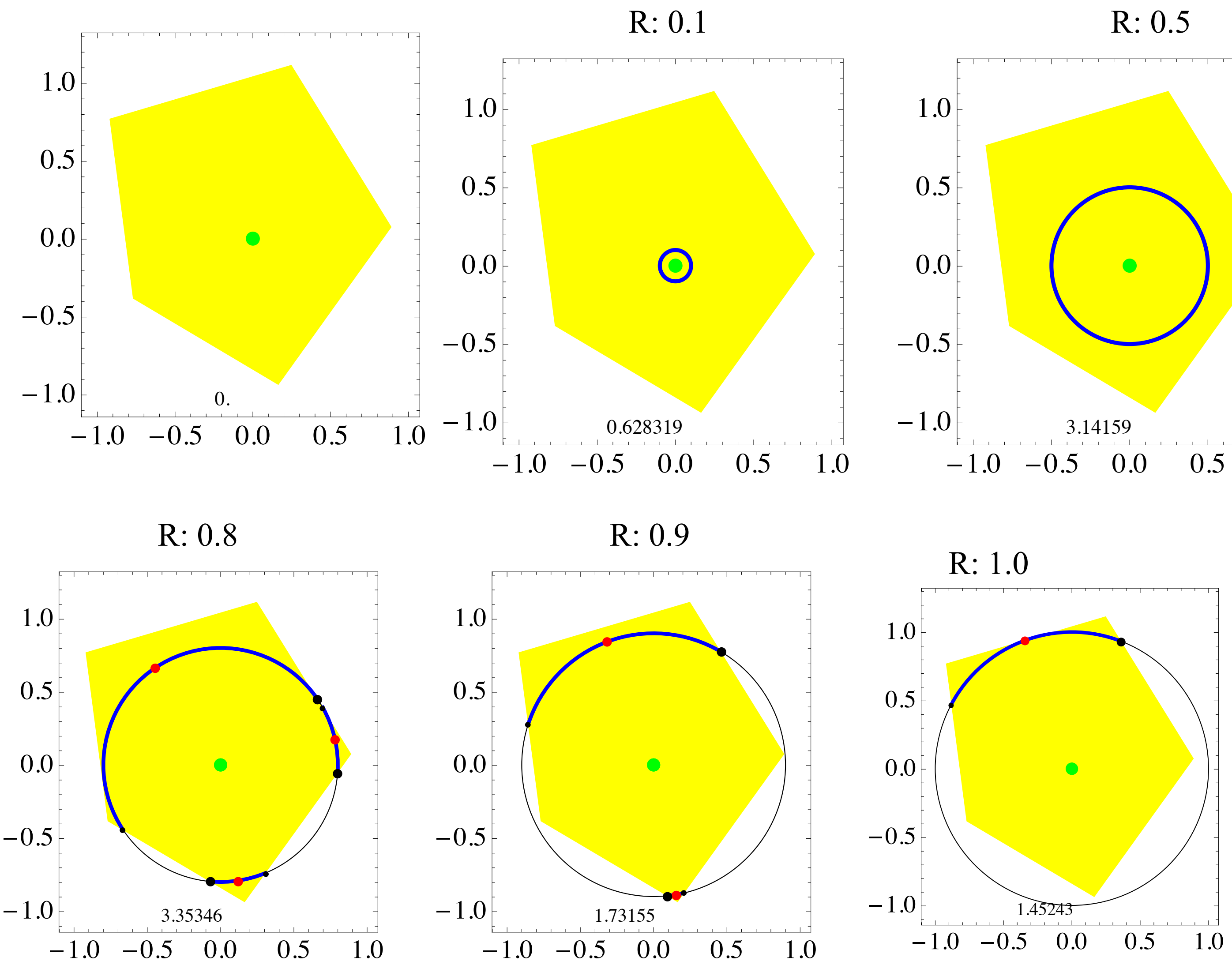
# Hazard Curve



# General areal sources

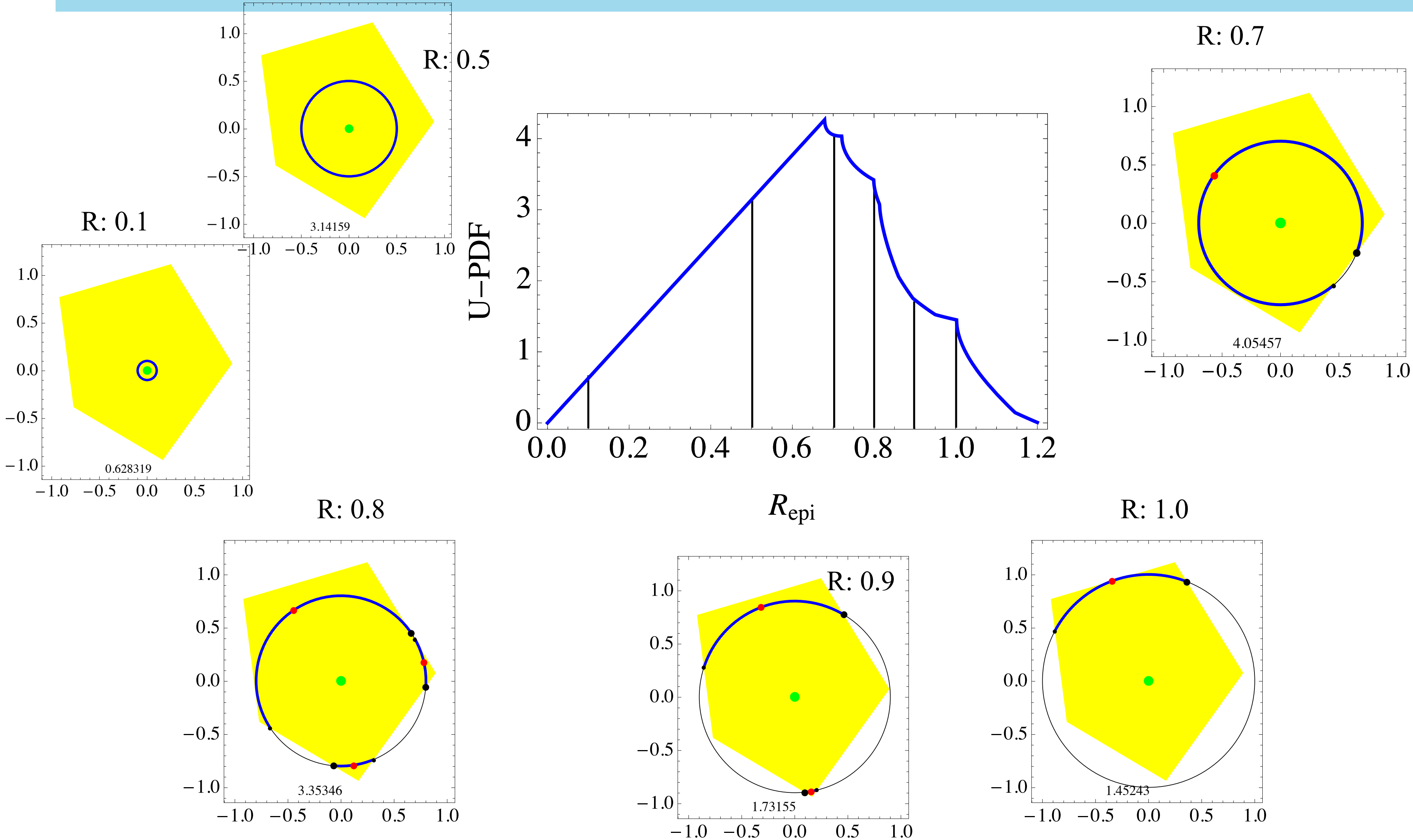

ArcLenghtInPolygon

# Distance distribution for general areal sources



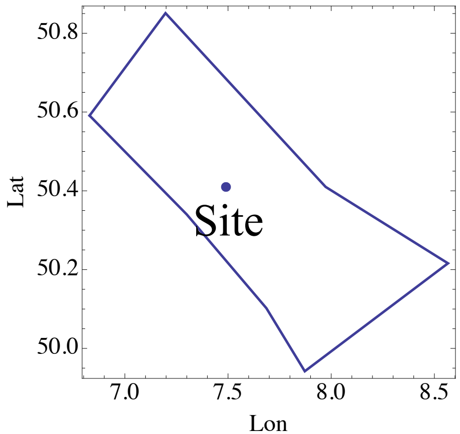

**Distance distribution**

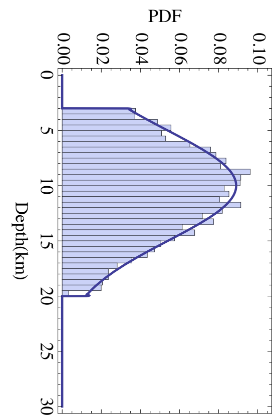
# Distance distribution for general areal sources

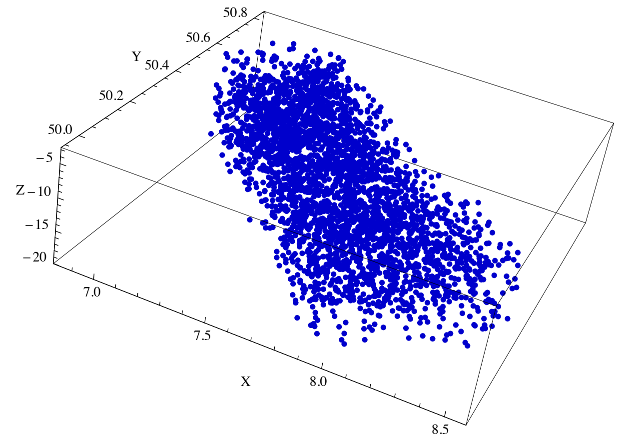


EpidistDistribInPoly

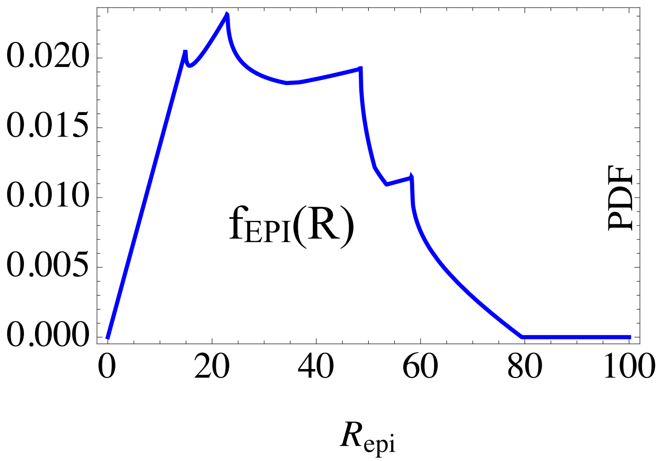
# Full distance distribution

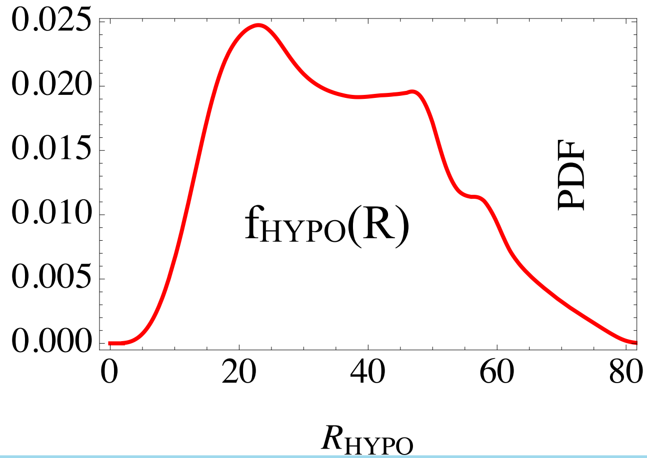


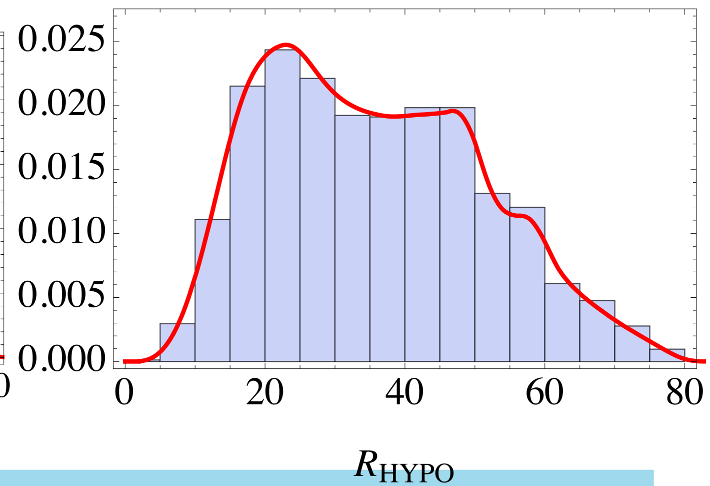




Comparison with simulation



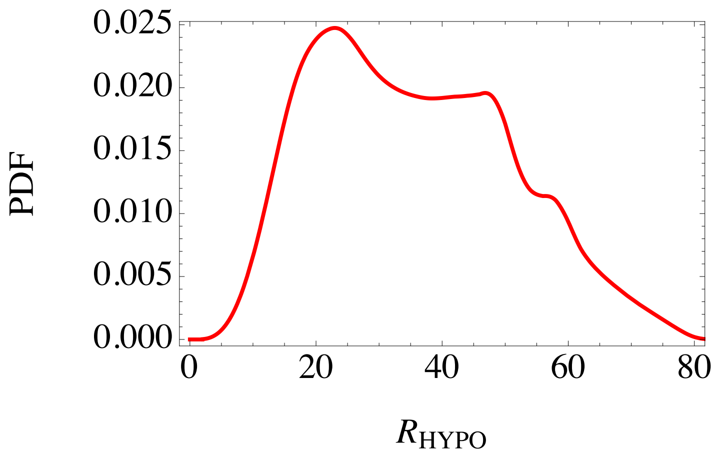
# Hazard integral

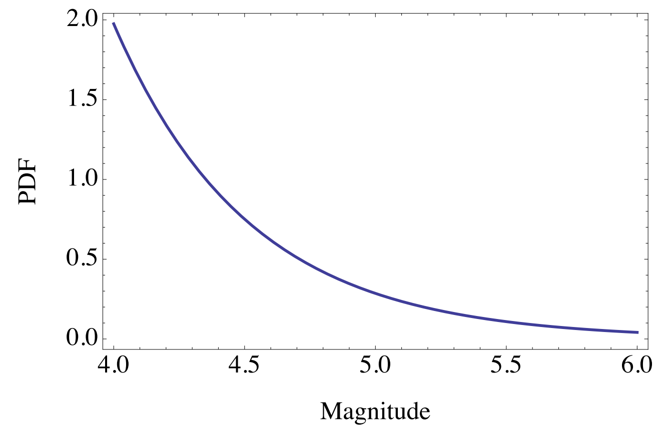
$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} PDF_{\ln(gm)}(\varepsilon | R, M) \cdot f_R(R) \cdot f_m(M) \cdot d\varepsilon \cdot dR \cdot dM$$

Ground motion model

Distance distribution

Magnitude distribution

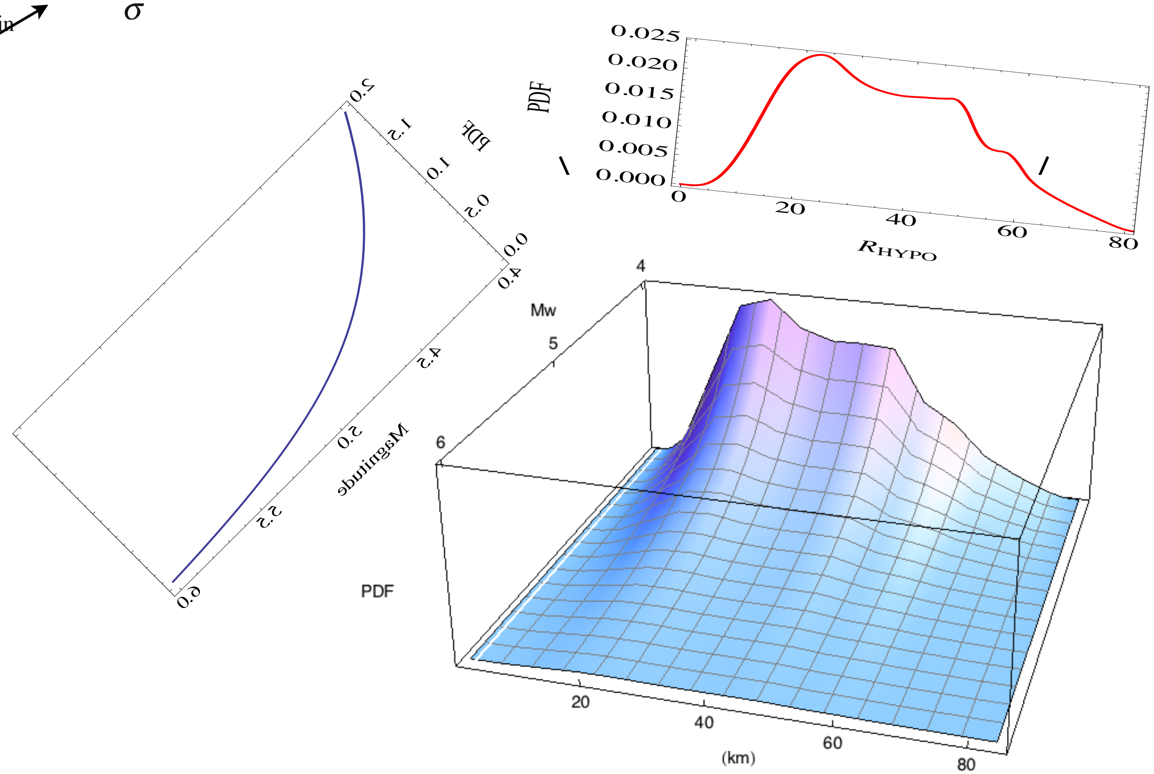


# Hazard integral

$$\nu(\mathrm{Sa} > \mathrm{a}) = \sum_{i=1}^{n_{\mathrm{sources}}} \mathrm{N_i}(\mathrm{M_{min}}) \cdot \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} \int_{\mathrm{R_{min}}}^{\mathrm{R_{max}}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \mathrm{PDF}_{\ln(gm)}(\varepsilon | \mathrm{R},\mathrm{M}) \cdot \mathrm{f_R}(\mathrm{R}) \cdot \mathrm{f_m}(\mathrm{M}) \cdot d\varepsilon \cdot d\mathrm{R} \cdot d\mathrm{M}$$

Ground motion model





Frank Scherbaum

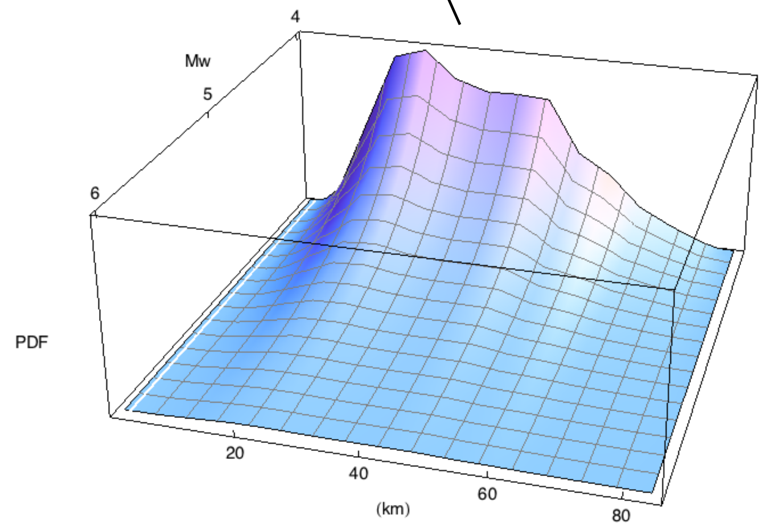
Monday, January 21, 13

# Hazard integral

$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} PDF_{\ln(gm)}(\varepsilon | R,M) \cdot \underbrace{f_R(R) \cdot f_m(M)} \cdot d\varepsilon \cdot dR \cdot dM$$

Ground motion model

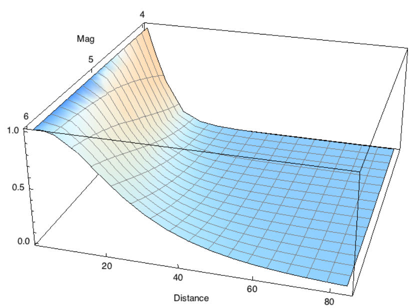
$1\ m/s^2$



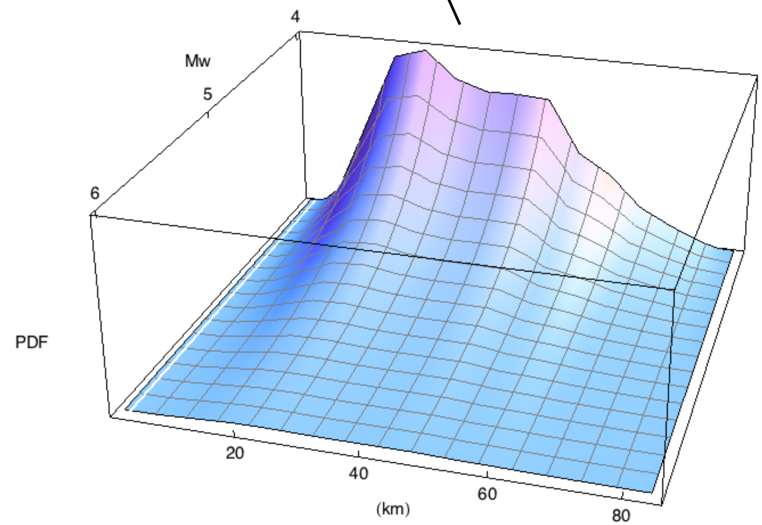
# Hazard integral

$$\nu(\mathrm{Sa} > \mathrm{a}) = \sum_{i=1}^{n_{\mathrm{sources}}} \mathrm{N_i}(\mathrm{M_{min}}) \cdot \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} \int_{\mathrm{R_{min}}}^{\mathrm{R_{max}}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \mathrm{PDF}_{\ln(gm)}(\varepsilon | \mathrm{R,M}) \cdot \underbrace{\mathrm{f_R}(\mathrm{R}) \cdot \mathrm{f_m}(\mathrm{M})}\cdot d\varepsilon \cdot \mathrm{dR} \cdot \mathrm{dM}$$

Ground motion model

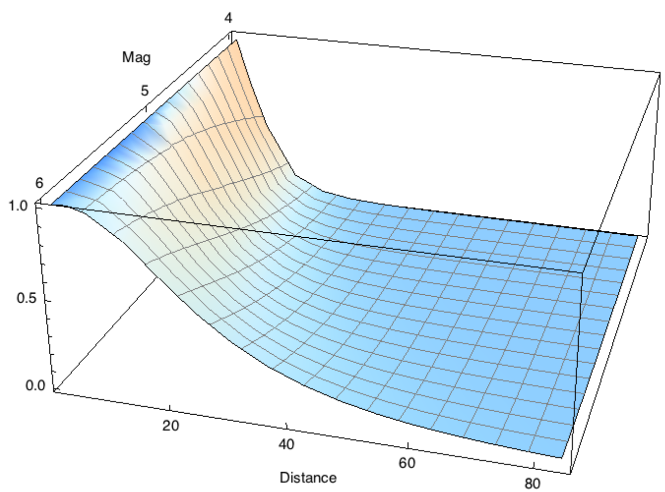


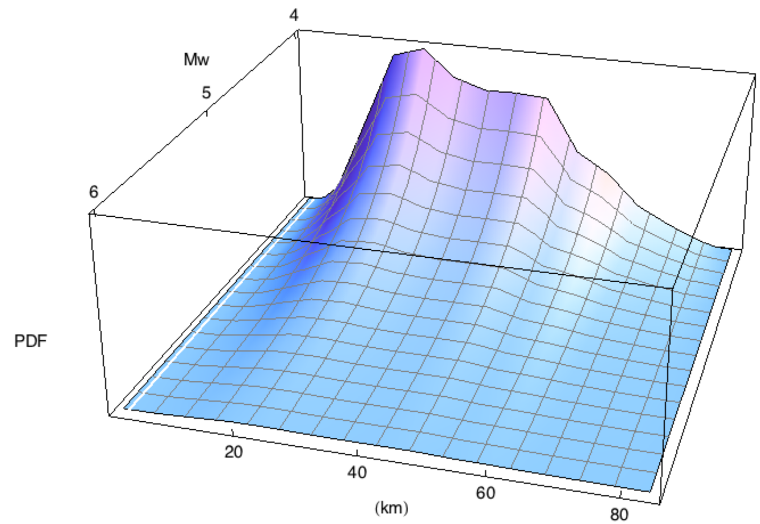

$1\ \mathrm{m/s^2}$



# Hazard integral

$$\nu(Sa > a) = \sum_{i=1}^{n_{sources}} N_i(M_{min}) \cdot \int_{M_{min}}^{M_{max}} \int_{R_{min}}^{R_{max}} \int_{\frac{\ln(a)-\ln(\mu)}{\sigma}}^{\infty} \underbrace{PDF_{\ln(gm)}(\varepsilon|R,M)} \cdot \underbrace{f_R(R) \cdot f_m(M)} \cdot d\varepsilon \cdot dR \cdot dM$$



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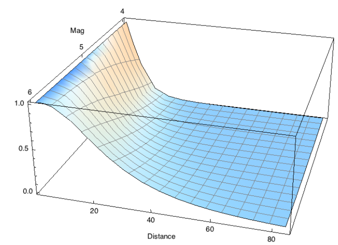


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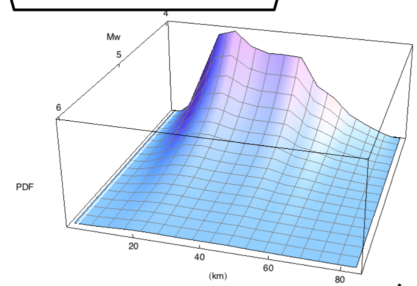
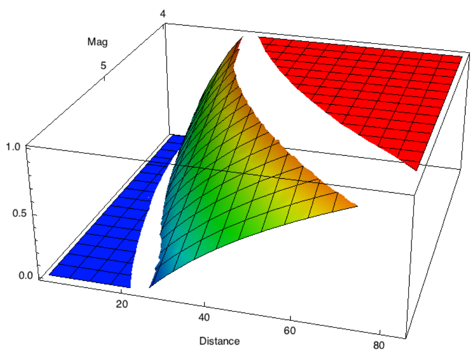
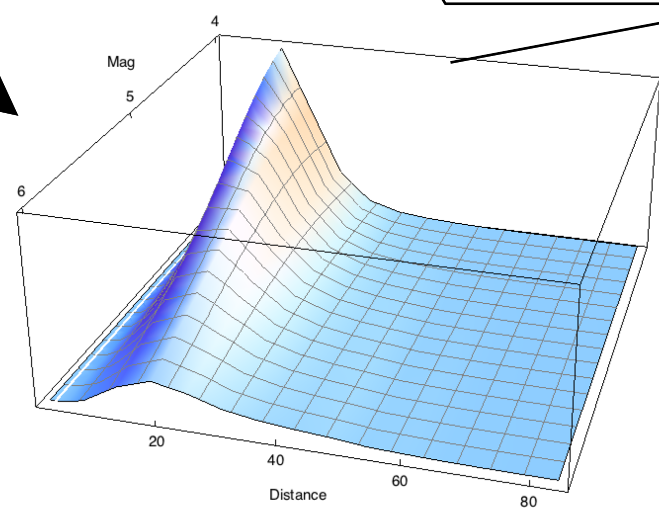



Corresponding distribution of ε




use ε for shading



# Hazard integral

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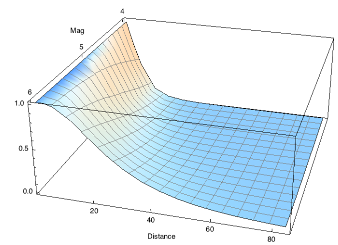


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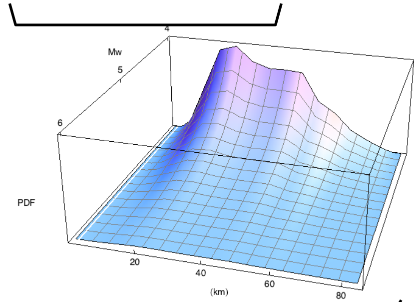
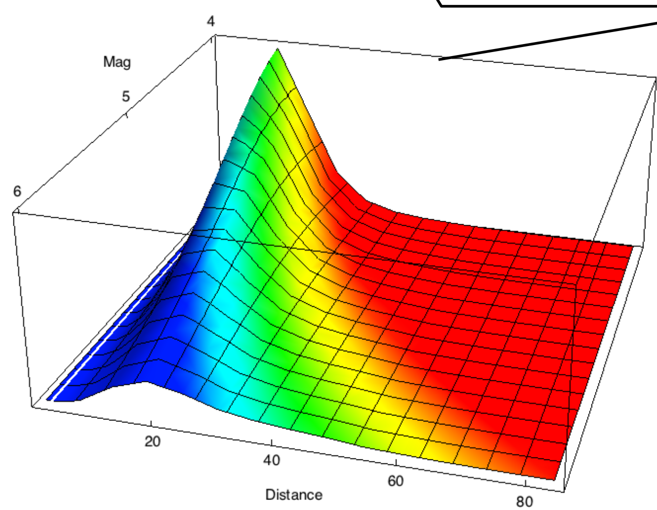
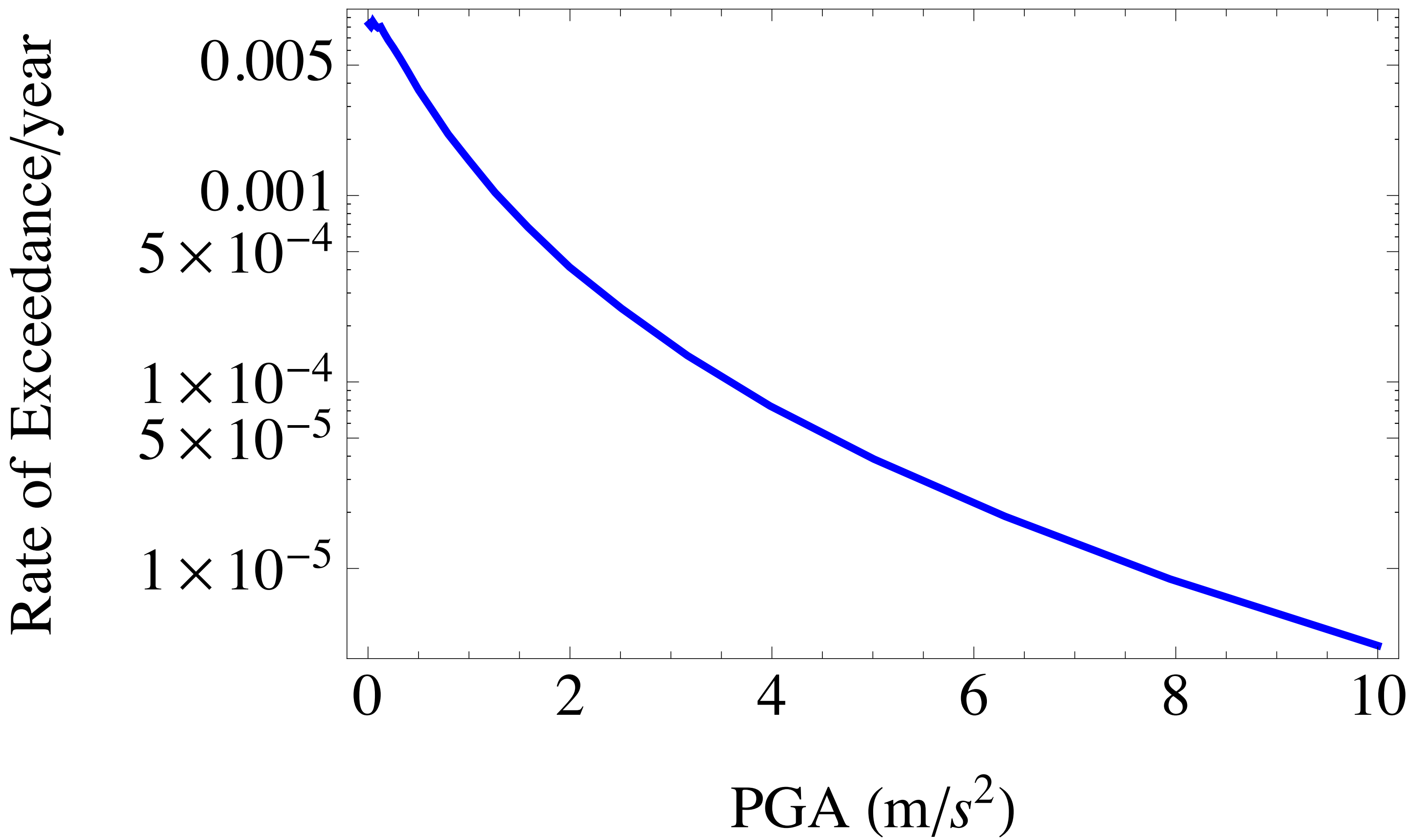



The value of the integral = volume under the surface

# Hazard curve




HazardIntegral